The Multicommodity Traveling Salesman Problem

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1 Introduction

The Minimum Latency Problem (MLP) [4] [9] [6] [3], also called the Traveling Repairman Problem or the Deliveryman Problem, is a variant of the Traveling Salesman Problem (TSP) [5] in which a repairman is supposed to visit the nodes of a graph in a way to minimize the overall waiting times of the customers located in the nodes of the graph. The problem was introduced and related to the TSP in 1967, by Conway, Maxwell and Miller [4], when the MLP was known as a type of scheduling problem. According to Goemans and Kleinberg [8], despite the obvious similarities to the classical TSP, the MLP appears to be much less well-behaved from a computational point of view.

We are presenting in this article, another variant of these problems that encompasses with a cost objective function, both the original TSP and the MLP. We are talking about the Multicommodity Traveling Salesman Problem (MTSP), where, at each node, a salesman delivers a demand $d_k \geq 1$ of a commodity that is specific to each city k. As in the original TSP, the traveling salesman pays the standard fixed cost to pass in an arc (i, j), but now he also faces a variable cost for each kind of commodity that needs to be carried across that arc. Each variable cost is proportional to the quantity of the correspondent commodity, in such a way that differences in both the unitary arc costs and in the quantities to be delivered imply differences of the operational costs to serve the customer nodes. In this extended TSP the weight of the salesman's vehicle is an important part of the total cost in a road, in a way that should be attended with some priority for customers with higher quantities and/or higher single transportation costs of the associated commodities. The MTSP joins on an unique problem the characteristics of the original Traveling Salesman Problem [5] and of a weighed formulation of the Single Vehicle Delivery Problem (SVDP) [2]. Since the SVDP is an extension of the classical MLP, the MTSP also encompasses the MLP.

2 The Multicommodity Traveling Salesman Problem

Consider a directed connected graph $G(V, E)$, where V denotes the set of nodes (cities) and E is a collection of arcs (roads). Suppose we have an origin node o and a set of nodes K, where $K = V - o$ and, for each node $k \in K$, a demand d_k of a specific commodity k should be delivered during a traveling salesman's tour. Suppose that the traveling salesman pays the standard fixed cost to traverse arc(i, j), but that he also faces a variable cost for each kind of commodity that needs to be carried across that arc. The objective is to deliver all the commodity demands by a tour that minimizes the sum of the fixed and variable flow costs.

For this problem we can define a mixed-integer linear programming formulation with the following set of variables: $x_{ij} = 1$ if the salesman travels across arc (i, j) and 0 otherwise;f_{ijk}: flow of commodity k transported across arc (i, j) with destination to demand node k. g_{ij} is the total aggregate flow through arc (i, j). The model has a following set of parameters: b_{ij} is the fixed cost paid by the salesman to travel in arc (i, j) and c_{ijk} is the unitary flow cost to transfer commodity k across arc (i, j).

The mathematical model is:

$$\min \sum_{(i,j)\in E} (b_{ij}x_{ij} + \sum_{k\in K} c_{ijk} f_{ijk}) \quad (1)$$

suject to

$$- \sum_{(o,j)\in E} f_{ojk} = -d_k \quad \forall k \in K \quad (2)$$

$$\sum_{(i,k)\in E} f_{ikk} = d_k \quad \forall k \in K \quad (3)$$

$$\sum_{(i,j)\in E} f_{ijk} - \sum_{(j,l)\in E} f_{jlk} = 0 \quad \forall k \in K, j \neq k \quad (4)$$

$$f_{ijk} \leq d_k x_{ij} \quad \forall (i,j) \in E, \forall k \in K \quad (5)$$

$$f_{ijk} \geq 0 \quad \forall (i,j) \in E, \forall k \in K \quad (6)$$

$$\sum_{(i,j)\in E} x_{ij} = 1 \quad \forall j \in V \quad (7)$$

$$\sum_{(i,j)\in E} x_{ij} = 1 \quad \forall i \in V \quad (8)$$

$$x_{ij} \in \{0,1\} \quad \forall (i,j) \in E \quad (9)$$

$$- \sum_{(o,j)\in E} g_{oj} = -\sum_{k\in K} d_k \quad \forall k \in K \quad (10)$$

$$\sum_{(i,k)\in E} g_{ik} - \sum_{(k,j)\in E} g_{kj} = d_k \quad \forall k \in K \quad (11)$$

$$g_{ij} \leq \sum_{k\in K} d_k x_{ij} \quad \forall (i,j) \in E \quad (12)$$

$$g_{ij} \geq 0 \quad \forall (i,j) \in E \quad (13)$$

The objective function (1) sums the costs for all the arcs of the network, with two parts for each arc. A first part refers to the fixed cost of traveling in the arc. The second part refers to the total flow charges associated with the transference, across the traveled arc, of all commodities, from the source node to the specific demand nodes. The separability of the objective function in both the arcs and the commodities is a clue for our decomposition strategy to solve such a large-scale problem.

We can observe that the constraints (10), (11), (12) and (13) resume information on flows and are the key to prevent the formation of cycles. With respect to the traditional formulation of Dantzig, Fulkerson and Johnson [5], that is limited to the space of the x_{ij} variables, the inclusion of the g_{ij} and f_{ijk} flow variables increases in polynomial form the number of variables of the problem. Instead of working with only one binary variable for each arc (i,j), the formulation (1)-(13) also operates with $\mid V \mid$ continuous variables for each arc (i,j), but this eliminates the need to include an exponential number of cycle elimination constraints, as it is the case in the original TSP model [5]. It is convenient to point out that the problem is not symmetrical, so existing two variables x_{ij}, two variables g_{ij} and two variables f_{ijk} for each connected pair of nodes [10].

3 Benders Decomposition for the MTSP

Benders partitioning method was published in 1962 [1] and was initially developed to solve mixed-integer programming problems. The computational success of the method to solve large scale multicommodity distribution system design models has been confirmed since the pioneering article of Geoffrion and Graves [7]. Now we will specialize the method for our model (1)-(13).

3.1 Problem Manipulation

A Benders partitioning method essentially relies on a *projection* problem manipulation, that is then followed by the solution strategies of dualization, outer linearization and relaxation. From the viewpoint of mathematical programming we can conceive a projection of our problem onto the space of *topological* variables x, thus resulting in the following implicit problem to be solved at a superior level:

$$\min_{x \in X} \sum_{(i,j) \in E} b_{ij} x_{ij} + t(x) \tag{14}$$

where $X = [x \mid$ for *x* fixed there is a feasible flow *f* satisfying (2)-(6)] and where $t(x)$ is calculated by the following problem to be solved at an inferior level:

$$t(x) = \min_{f \geq 0} \sum_{(i,j) \in E} \sum_{k \in K} c_{ijk} f_{ijk} \ \text{ suject to } (2) - (5) \text{ for } x \text{ fixed.} \tag{15}$$

The flow feasibility requirement related to a topological variable $x \in X$ implies that the components for which $x_{ij} = 1$ composes a ring rooted at the origin o and destined to every demand node $k \in K$. At the inferior level, the right-hand size of the primal problem (15) is dependent on the x values, but the feasible solution set of the corresponding dual problem is always the same for any fixed x. With the use of previously generated extreme points of this constant dual set, it is possible, at a superior level of each Benders cycle, to have a better underestimation of the operational cost related to any topology x. The idea is to choose at each cycle h a solution x^h that minimizes the sum of the known fixed cost $\sum_{(i,j) \in E} b_{ij} x_{ij}$ plus the best known underestimation of the operational cost related to the topology x^h. The method combines the use of dualization, outer linearization and relaxation in such a way to approximate the project problem (14). We will analyze now the subproblem to provide further detail on the choices made.

3.2 Subproblems

For a fixed cycle C^h, associated with the vector x^h, the computation of a minimal cost flow $t(x^h)$ can be separated in a series of trivial network problems. Let P^h_{ok} be the set of arcs in the path, from the source node to the demand node k, that has been defined by the master problem of cycle h. We now state in detail the primal-dual pair to be solved for each commodity $k \in K$.

3.2.1 Primal and Dual subproblems for commodity k when x=x^h

$$\min \sum_{(i_j) \in E} c_{ijk} f^h_{ijk} \tag{16}$$

$$\text{subject to } -\sum_{(o,j) \in E} f^h_{ojk} = -d_k, \ \forall k \in K \tag{17}$$

$$\sum_{(i,k) \in E} f^h_{ikk} = d_k, \ \forall k \in K \tag{18}$$

$$\sum_{(i,j) \in E} f^h_{ijk} - \sum_{(j,l) \in E} f^h_{jlk} = 0, \ \forall k \in K, j \neq k \tag{19}$$

$$f^h_{ijk} \leq d_k x^h_{ij}, \ \forall (i,j) \in E, \forall k \in K \tag{20}$$

$$f^h_{ijk} \geq 0, \ \forall (i,j) \in E, \forall k \in K \tag{21}$$

The trivial and unique solution of the problem is: $f^h_{ijk} = 1$ if (*i*,*j*) $\in P^h_{ok} \subseteq C^h$ and $f^h_{ijk} = 0$ otherwise.

The dual problem associated to the subproblem given by the objective function (16) and constraints (17)-(21) is:

$$\max_{\rho^h,\alpha^h \geq 0} d_k(\rho^h_{kk} - \rho^h_{ok} - \sum_{(i,j)\in E} x^h_{ij}\alpha^h_{ijk}) \tag{22}$$

$$\text{subject to } \rho^h_{jk} - \rho^h_{ik} - \alpha^h_{ijk} \quad \leq \quad c_{ijk}, \ \forall k \in K \tag{23}$$

This dual problem has many feasible solutions in contrast with the primal problem that has an unique trivial solution. Since $f^h_{ijk} = d_k > 0, \forall (i,j) \in P^h_{ok} \subseteq C^h$ we have from the complementary slackness condition that: $\rho^h_{jk} - \rho^h_{ik} - \alpha^h_{ijk} = c_{ijk}, \forall (i,j) \in P^h_{ok} \subset C^h$

in such a way that we can construct, associated with the primal solution x^h, the following dual feasible solution:

$$\begin{aligned}
\rho^h_{ok} &= 0, \ \forall k \in K, \quad \text{for the origin } o, && (24)\\
\rho^h_{jk} &= \rho^h_{ik} + c_{ijk}, \ \forall (i,j) \in P^h_{ok} \subset C^h, && (25)\\
\alpha^h_{ijk} &= 0, \ \forall (i,j) \in P^h_{ok} \subset C^h, && (26)\\
\alpha^h_{ijk} &= \rho^h_{jk} - \rho^h_{ik} - c_{ijk}, \ \forall (i,j) \in E - C^h \text{ such that } \rho^h_{jk} - \rho^h_{ik} > c_{ijk}, && (27)\\
\alpha^h_{ijk} &= 0, \ \forall (i,j) \in E - C^h \text{ such that } \rho^h_{jk} - \rho^h_{ik} \leq c_{ijk}, && (28)
\end{aligned}$$

The systematic evaluation of the dual variables with commodity meaningful values is the clue for an efficient implementation. Here the two series of dual variables can be interpreted as price information. Each variable ρ^h_{ik} represents the price of the establishment of the communication $k(k \in K)$ from the origin node o to node $i(i \in V)$ in cycle $h(h = 1, ..., H)$. On the other hand, each variable α^h_{ijk} gives for commodity k the value of an additional unit of capacity at arc(i, j). The dual variable α^h_{ijk} evaluates for commodity k the maximal reduction in the operational cost that could be gained with the introduction of arc (i, j). In the case of transportation systems, it can also be understood as a tax to be paid with the use of arc(i, j) in order to maintain the distribution agents with no positive profit.Remark that the constant dual solution set (23) represents spacial prices for which ones there is no positive profit for any distribution agent that pays the cost c_{ijk} to flow commodity k through arc(i, j) [10].

3.3 Master Problem

The mathematical model of the master problem is constituted by the following objective function:

$$\min_{x\in X} \sum_{(i,j)\in E} b_{ij}x_{ij} + t \tag{29}$$

subject to the constraints (7)-(13) and by the Benders cut constraint:

$$t \geq \sum_{(i,j)\in E} d_k(p^h_{kk} - \sum_{(i,j)\in E} \alpha^h_{ijk}x_{ij}) \ \ h = 1, ..., H \tag{30}$$

The parameter h is a cycle counter and indicates the number of Benders cuts that must be taken into account. For given h and k, the corresponding value in the right-hand-side of constraints(15) provides a lower bound on the cost of the flow that leaves the origin node to the demand node k. The variable t that appears in objective function (29) is the best known lower bound on the total operational cost.

Table 1: Table of Results

Problem	$\|V\|$	$\|E\|$	θ	LR gap (%)	FC/VC (%)	CPLEX Time (s)	CPLEX gap (%)	Benders Cycles	Benders Time (s)	Benders gap (%)	Cycle 1 gap (%)
P15A	15	28	0.5	27.53	85.302	4.57	0	8	3.22	0	0
P15A2	15	28	0.1	15.46	249.54	4.06	0	7	3.25	0	3.416
P20A	20	43	0.5	31.469	106.271	44.78	0	39	246.80	0	1.7541
P20A2	20	43	0.1	22.144	184.993	28.84	0	29	130.95	0	0
P25A	25	50	0.1	21.048	239.971	110.70	0	12	59.01	0	0
P26A	26	68	0.1	23.829	162.927	8377.03	13.27	74	7357.07	0	0
P28A	28	53	0.1	23.573	245.837	68.70	0	10	30.62	0	2.871
P28B	28	57	0.2	24.607	144.056	340.21	0	74	2088.56	0	0
P28B2	28	57	0.1	21.013	177.939	336.69	0	69	1685.60	0	0
P29A	29	54	0.1	23.388	186.642	266.89	0	42	553.81	0	1.067
P29B2	29	54	0	0	-	2.54	0	1	1.27	0	0
P30A	30	57	0.1	17.497	232.654	96.81	0	12	49.13	0	0
P32A	32	50	0.1	27.792	173.237	95.81	0	13	39.77	0	0
P34A	34	56	0.1	25.278	181.768	190.34	0	9	41.47	0	0.143
P36A	36	62	0.1	29.186	173.416	33.92	0	3	2.32	0	0.812
P36B	36	61	0.1	32.134	124.575	429.29	0	55	1114.74	0	2.894
P38A	38	62	0.1	27.686	171.307	276.91	0	24	208.20	0	0
P40A	40	73	0.1	29.965	151.708	1294.61	0	37	990.24	0	0
P40B	40	70	0.1	31.137	159.656	101.21	0	5	10.81	0	0
P45A	45	84	0.1	36.754	129.092	221.75	0	3	20.25	0	0
P50A	50	92	0.1	35.805	121.206	2739.98	0	21	3468.72	0	2.336
P55A	55	105	0.1	39.538	114.352	2165.05	0	3	44.91	0	0
P55A2	55	105	0.2	54.220	57.176	1787.85	0	3	47.38	0	0
P60A	60	121	0.1	42.211	113.639	7202.45	24.06	7	3529.40	0	5.048
P65A	65	122	0.1	44.732	89.598	3074.04	0	17	565.91	0	0.048

4 Computational results

The tests were executed in a Sun Blade 100 computer that has a UltraSPARC processor of 500 MHz and 1 Gb of RAM memory. The operational system is the Solaris 5.8. The Benders decomposition algorithm was implemented in C++ using the library *Concert Technology 1.0* of CPLEX®7.0. In all the experiments we have solved the problems through two methods: the solver CPLEX®and the algorithm presented in the previous section, that uses the Benders decomposition method. The CPLEX®was used with the standard values for the parameters, except for the total time limit that was modified for 7200 seconds.

The values of b_{ij} related to the distances among the cities were chosen randomly between 1601 and 3200. This program generates random demands between 1 and 130 units $\forall i \in V - o$ and also generates variable costs between 0.1% and $\theta\%$ of the fixed cost related to each arc (i, j) and to each product k. The values of the experiments are shown in the table (1).

The field *LR gap (%)* gives the linear relaxation Gap from formulation (**??**)-(**??**). The field *Cycle 1 gap(%)* gives the gap, in porcentage, between the soluton given by the first Benders cycle and the optimal solution.

Table (1) shows that Benders decomposition was faster than CPLEX®in 18 of the 25 cases, mainly in those where relation (fixed cost/variable cost), *FC/VC*, is higher. It was verified that in some cases the Benders method was more than 10 times faster than the resolution executed entirely by CPLEX®.

Despite to know that the Benders decomposition method is somewhat efficient, perhaps the most important experiment of the table (1) is experiment P29B2, where the variable cost is zero and, consequently, the problem becomes the TSP. Due to the characteristics of the Benders algorithm in the previous section, it is not surprising that Benders solves the problem in the first cycle, since our algorithm has as result of the first Benders cycle the value of the TSP. The good surprise is related to the *LR gap (%)*, that is, the solution found for the CPLEX®for the relaxed problem (without the constraints of integrality).

5 Concluding Remarks

We have introduced a new problem that we call the Multicommodity Traveling Salesman Problem. Through the model (1)-(13) and the Benders decomposition method we have solved instances of up to 65 nodes. Despite the number 65 does not seems very significant we have to point out that for the Single Vehicle Delivery Problem (SVDP), Bianco and al. [2] have presented results for 30 nodes, half of the size of the results presented here for the MTSP. Besides, Benders decomposition was faster than CPLEX®in 18 of the 25 instances, another example of decomposition strategy to solve mixed integer linear problems with significant results.

From the solved problems we observed that as much as the instances are closer to the TSP, that is, the parcel relative to the variable cost is smaller, we can obtain more satisfactory results. This occurs both with the direct use of the software CPLEX®and also with the use of the solver as part of the Benders decomposition algorithm. We also evidence that, besides the fact that the two methods are more efficient when the problem approximates to the TSP, in this case Benders decomposition turns out to be many times faster than the direct use of the solver CPLEX®. Moreover, the problem becomes easier as much sparse is the graph $G = (V, E)$, that is, when reduces the number of routing options that the algorithms have to choose. Sparsity is good for both methods, but again we can obtain a bigger improvement with the Benders decomposition method.

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