Extended Node-Arc Formulation for the K-Edge-Disjoint Hop-Constrained Network Design Problem

Quentin Botton
Université catholique de Louvain, Louvain School of Management, (Belgique)
botton@poms.ucl.ac.be
Bernard Fortz
Université libre de Bruxelles, Département d'informatique, (Belgique)
bernard.fortz@ulb.ac.be

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Abstract

This paper considers the *K*-edge-disjoint hop-constrained Network Design Problem (*HCNDP*) which consists in finding a minimum cost subgraph such that there exists at least *K*-edge-disjoint paths between origins and destinations of demands, and such that the length of these paths is at most equal to a given parameter *L*. This problem was considered in the past using only design variables. Here, we consider an extended node-arc formulation, introducing flow variables to model the paths. We conjecture that our formulation leads to the complete description of the associated polyhedron and we provide an algorithm leading to good performances in terms of computing times.

1. Introduction

Our society is increasingly dependent on large-scale, networked information systems of remarkable scope and complexity. This dependency magnifies the far-reaching consequences of system damage from attacks and intrusions. Yet no amount of security can guarantee that systems will not be damaged. Incorporating survivability capabilities into a network can mitigate the risks. Here, survivability is the capability of a system to fulfill its mission in a timely manner despite intrusions, failures, or accidents. This concept of survivability allows networks to remain functional when links are severed or nodes fail, that is, network service can be restrored in the event of catastrophic failures.

Consider an undirected graph $G = (V, E)$, where V represents the node (vertex) set, and E the set of edges or potential links. To express the survivability conditions, we need to introduce the following graph-theoretic concepts. Given two distinct nodes o (the origin node of demand) and d (the destination node of demand) of V, an st-path is a sequence $P = (v_0, (v_s, v_1), v_1, ..., (v_{l-1}, v_l), v_l)$, where $l \geq 1$, $v_0, v_1, ..., v_l$ are distinct nodes, $v_0 = o$, $v_l = d$, and (v_{i-1}, v_i) is an edge connecting v_{i-1} and v_i (for $i = 1, ..., l$). A collection $P_1, P_2, ..., P_l$ of st-paths is called edge-disjoint (respectively node-disjoint) if any edge (respectively, any node except for o and d) appears in at most one path. A subgraph H of G is called (K-)edge-survivable (respectively, (K-)node-survivable) if for any $o, d \in V$, H contains at least a prespecified number K of edge-disjoint (respectively, node-disjoint) st-paths. Suppose that each edge $e \in E$ has an installation cost $c(e) \in \mathbb{R}_+$, then the edge-survivable network design problem, denoted by $ESNDP$, consists of finding a edge-survivable subgraph of G with minimum total cost, where the cost of a subgraph is the sum of the costs of its edges. Similarly, the node-survivable network-design problem, denoted by $NSNDP$, consists in finding a minimum-cost node-survivable subgraph of G

In general, the survivability requirement is not sufficient to guarantee a cost effective routing. Indeed, the alternative routing paths may be too long and then too much expensive to be suitable. In consequence, in order to guarantee a level quality of service, further technical constraints have to be added; in particular, one can impose a limit on the lenght of the routing paths, so that if one of the paths fails, the traffic may be rerouted on the second one with a quality of service guaranteed in terms of delay. In many works on the subject, the lenght of the routing path is considered as the number of links, (also called hops) used in the path, and then we talk about a hop-constrained path.

Survivability problems are generally tackled by two different rerouting techniques. The first one called local rerouting consists in rerouting the traffic between the extremities of the failed link. In this study we focus on the second technique called end-to-end rerouting. This technique considers all the demands affected by the failure and reroute these demands from their respective origins to their respective destinations. Here we focus on end-to-end rerouting with hop constraints. Three major problems of this kind have been studied before.

The hop-constrainded spanning tree problem consists in finding the minimum spanning tree such that the path between the root and any other node has no more than L hops. This problem has been largely studied by Gouveia [8], [9] in which the author proposes a multicommodity network flow formulation and discuss the use of lagragian relaxation to improve the lower bound. In [10], Gouveia and Requejo proposed the hop-indexed formulation using the layered representation of the graph. In [3] Dahl gives a polyhedral study of the problem when the number of hops is limited to 2.

The hop-constrained path problem consists on finding between two distinguished nodes s and t a minimum cost path with no more than L hops with L fixed. In [9], Gouveia proposed valid inequalities for the L-path polytope problem and called these inequalities st-cut inequalities. In [4], Dahl and in [6], Dahl and Gouveia give a complete description of the polytope when $L \leq 3$ using their jump-inequalities for which the separation problem can be solved in polynomial time. In [7], Dahl, Gouveia et al. extend the previous work when $L = 4$.

The third problem is the hop-constrained network design problem. This problem focuses on finding a minimum cost subgraph such that there are at least K edge-disjoint paths between each pair of terminals. Moreover, each path must use at most L hops. In [1], Balakrishnan et al. give a mixed integer formulation for the problem when $K = 1$, they derive a Lagrangean relaxation to improve the lower bound. In [12], Pirkul et al. propose multicommodity network flow models and some heuristics based on linear relaxation. In [5], Dahl et al. study the problem where there are for each pair of terminals K edge-disjoint paths of maximum lenght L. They prove that this problem is NP-hard even when $K = 1$ and $L = 2$. In [11], Huygens et al. study the same problem but when there is only one commodity, or one pair of terminals. They formulate an IP model when $K = 2$ and $L = 3$ in the space of the design variables. Bley [2] shows that the single demand node-disjoint lenght-restricted paths is NP-complete.

In this paper, we propose an extended formulation for the K-Edge-Disjoint Hop-Constrained Network Design Problem in the single commodity case. We conjecture that our formulations give a complete description of the polytope when $K = 2$ and we propose a polynomial time algorithm for this case.

2. Extended Formulations

In order to propose a model for the K-Edge-Disjoint Hop-Constrained Network Design Problem in the single commodity case, we use the layered representation proposed by Gouveia in [10]. This representation transforms the original graph by taking into account the hop-constraints. The idea is to duplicate all the nodes $L - 1$ times when L is the maximum number of hops. This process creates exactly L different layers as it is illustrated in Figure 1.

From the original non-directed graph $G = (V, E)$ we create a directed layered graph $G^{'} = (V^{'}, A^{'})$ where $V^{'}$ represents the set of nodes and where $A^{'}$ is the set of directed arcs in the layered graph. $V^{'} = \{V_1^{'} \cup V_2^{'} \cup ... \cup V_{L+1}^{'}\}$ where $V_1^{'} = \{o\}$, $V_{L+1}^{'} = \{d\}$ and $V_l^{'} = V \backslash \{o\}$, $l = 2, ..., L - 1$. The set of arcs is defined by $A^{'} = \{\{(i, j) \mid ij \in E \ and \ i \in V_l^{'} \ and \ j \in V_{l+1}^{'}, l = 1, ..., L\} \cup \{(i, j) \mid ji \in E \ and \ i \in V_l^{'} \ and \ j \in V_{l+1}^{'}, l = 1, ..., L\}\}$. Note that each path between o and d in the layered graph is made of a maximum of L

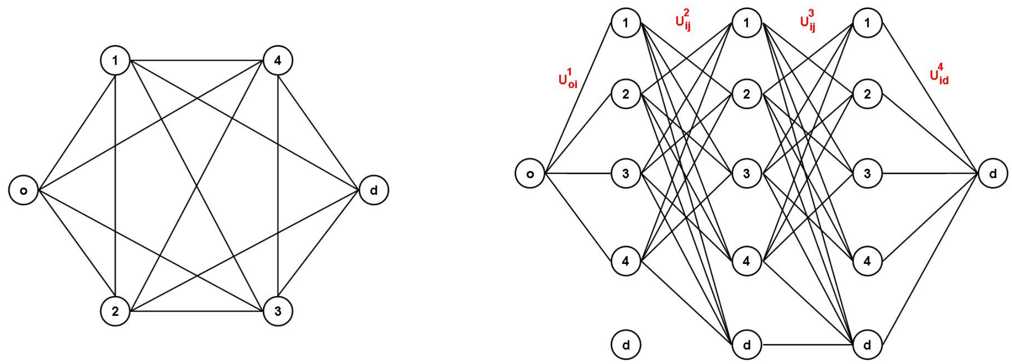

Figure 1: Basic Network and its layered representation when $L = 4$

arcs (hops). Linked to this layered representation we have the following model:

$$\min \sum_{l \in L} \sum_{(i,j) \in A} c_{ij} U^l_{ij} \tag{1}$$

subject to

$$\sum_{j:(j,i) \in A} U^l_{ji} - \sum_{j:(i,j) \in A} U^{l+1}_{ij} = \begin{cases} -K & if \quad (i = o) \; and \; (l = 0) \\ K & if \quad (i = d) \; and \; (l = L) \\ 0 & else \end{cases} \quad , \quad \forall i \in N, \forall l \in L, \tag{2}$$

$$U^l_{ij} \in \{0, 1\}, \qquad \forall (i,j) \in A, \forall l \in L. \tag{3}$$

In this model U^l_{ij} represents the quantity of information send on arc (i, j) when arc (i, j) is the lth hop in the path. The parameter c_{ij} represents the cost to send one unit of information on arc (i, j). (1) represents the objective fonction which minimizes the total cost when (2) are simply the flow conservation constraints. To model the disjoint paths we send from the origin node o a quantity equal to the number of disjoint paths K and we add the constraint (3) which is an upper bound on the flow variables. When $K = 1$ this formulation is similar to the one proposed by Dahl et al. in [7]. Note that this formulation has an integral linear relaxation as it is a network flow problem.

When $K > 1$, a flow of value K in the layered graph corresponds to a solution of our problem if and only if the transposition of the solution in the original graph uses each arc at most once. This remark leads to add a new constraint in our model:

$$\sum_{l \in L} U^l_{ij} \quad \leq 1, \quad \forall (i,j) \in A \tag{4}$$

Our numerical experiments lead us to conjecture that the addition of (4) does not break integrality of the formulation, as the linear relaxation was integral for all the instances we tested. As a first step towards a proof, we propose a Lagrangean relaxation approach to solve the problem. We are currently working on a proof that this algorithm converges in polynomial time to a feasible solution of the problem whith a Lagrangean cost equal to the original cost, therefore proving the integrality of our formulation.

3. Algorithm

Our algorithm is based on a Lagrangean relaxation (called here LR model) based on the relaxation of constraints (4) with multipliers $\overline{\beta_{ij}}$. The Lagrangean subproblem can be written as:

$$\min \sum_{l \in L} \sum_{(i,j) \in A} (c_{ij} + \overline{\beta_{ij}}) U_{ij}^l - \sum_{(i,j) \in A} \overline{\beta_{ij}} \tag{5}$$

subject to

$$\sum_{j:(j,i) \in A} U_{ji}^l - \sum_{j:(i,j) \in A} U_{ij}^{l+1} = \begin{cases} -K & if \quad (i = o) \; and \; (l = 0) \\ K & if \quad (i = d) \; and \; (l = L) \\ 0 & else \end{cases} \quad , \quad \forall i \in N, \forall l \in L, \tag{6}$$

$$U_{ij}^l \leq 1, \qquad \forall (i,j) \in A, \forall l \in L, \tag{7}$$

$$U_{ij}^l \geq 0, \qquad \forall (i,j) \in A, \forall l \in L. \tag{8}$$

The Lagrangean subproblem computes K-edge disjoint shortest paths in the layered network. This work can be done by using e.g. the algorithm of Suurballe [13]. The Lagrangean multipliers $\overline{\beta}$ are increased when an arc in the original network is used more than once. If for some $(i,j) \in A$, (4) is not satisfied, we compute the length of the shortest path with at most L hops not using this arc, and we increase $\overline{\beta_{ij}}$ by the difference between this length and the length of the shortest path in the current solution.

For $K = 2$, we conjecture that this approach converges in polynomial time to an optimal solution of our problem.

4. Future work

We are currently working on the proof of convergence of our algotithm for $K = 2$ and for any value of L, and the proof of the integrality of the extended formulations. We will also extend this work to the multicommodity case. Because the size of the model grows rapidly when the number of commodities increases, we will experiment Benders decomposition on this model.

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