Assignments, Matchings, and the European Roots of Combinatorial Optimization

EURO Gold Medal 2018 Laureate Lecture

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1. The Fifties: Birth of Combinatorial Optimization

2. The Assignment Problem and Kőnig's Matching Theorem

3. The First Egerváry's Theorem (1931) and Strong Duality

4. Jacobi (1851) and the Hungarian method

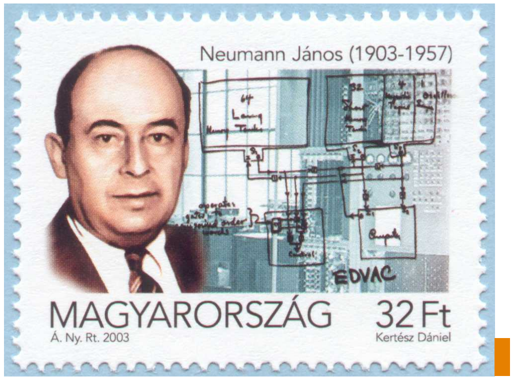
**5. Open shop, satellite communications,
and the Second Egerváry's Theorem**

The birth of Combinatorial Optimization

- After the milestones of the **Forties**,

  - **1947: Simplex Algorithm G.B. Dantzig**

  - **1948: LP Duality A.W. Tucker**

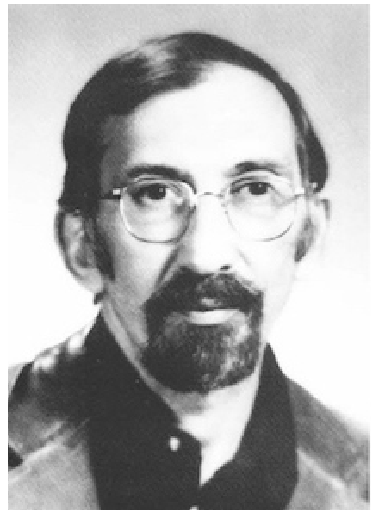


$\Leftarrow$ ... **1947 John Von Neumann**

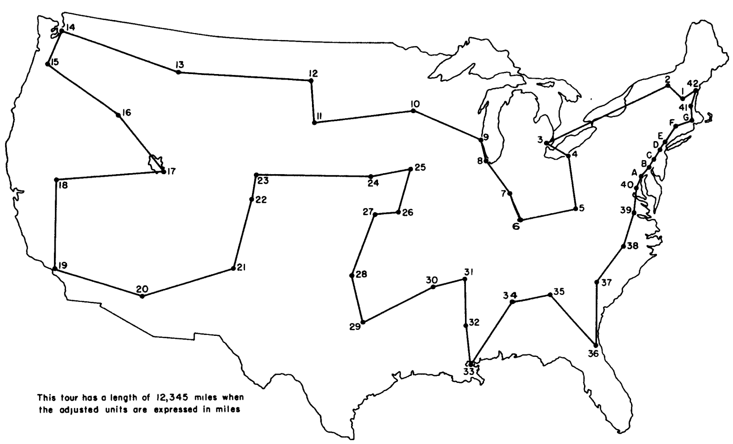
- the **Fifties** saw the birth of modern Combinatorial Optimization.

The Fifties: Specific results

- **1954: Traveling Salesman Problem** (the 49-city instance):

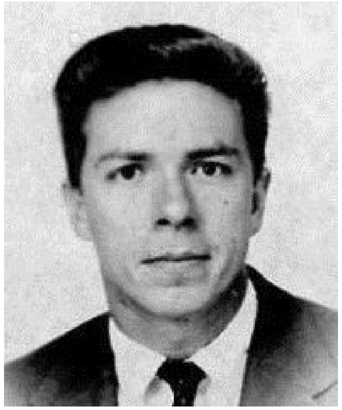

G.B. Dantzig, D.R. Fulkerson, and S.M. Johnson:



ILP formulation,
cutting planes

The Fifties: Specific results

- 1955: Assignment Problem: H.W. Kunh

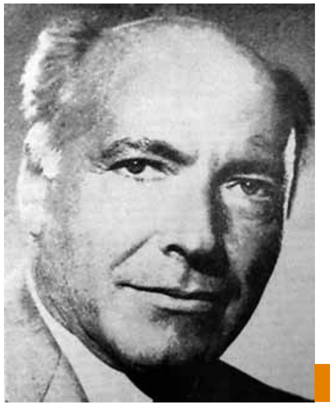


Hungarian algorithm

⇑ ⇑

D. Kőnig, 1916

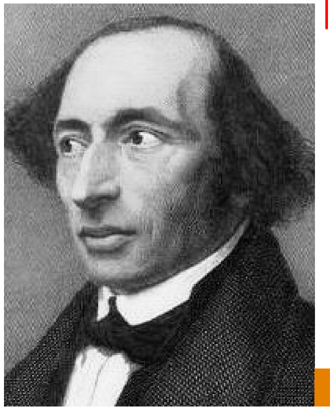
J. Egerváry, 1931



⇕ ⇑

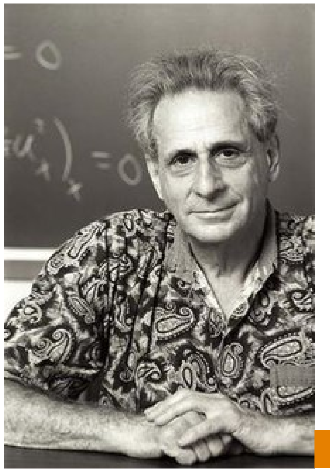
Frobenius, 1917,1912

. . . Jacobi, 1851



The Fifties: Specific results

- **1956: Minimum Spanning Tree:**

R.C. Prim  J.B. Kruskal, Jr. 

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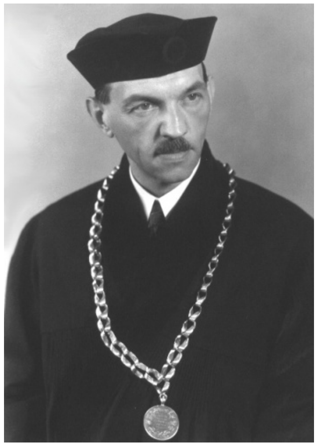
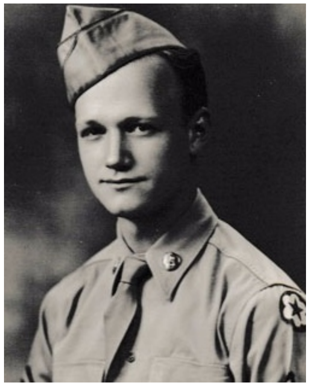
V. Jarník, 1930  O. Borůvka, 1926 

The Fifties: Specific results

- **1956: Maximum flow:**
  Max-flow min-cut



L.R. Ford, Jr.

D.R. Fulkerson

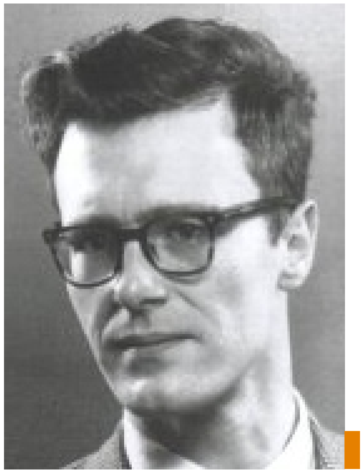
⇑

K. Menger, 1927
Fundamental theorem on connectivity and disjoint paths

. . . **D. Kőnig, 1916**

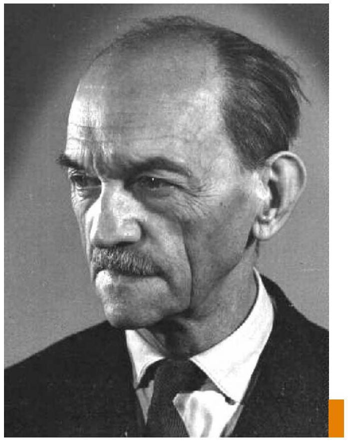
The Fifties: Specific results

- **1956-1959: Shortest paths:**

L.R. Ford, Jr.  E.W. Dijkstra  et al.

⇕ ⇕

K. Menger, 1927
V. Jarník, 1930

The Fifties: General methodologies

- **1953: Dynamic programming R.E. Bellman**

- **1958: Cutting plane algorithm: R.E. Gomory**
- **1960: Branch-and-Bound method for the ILP:**

Ailsa Land  **Alison Doig** 

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The Assignment Problem (AP)

- Given an $n \times n$ matrix $C = (c_{ij})$, find a **permutation** φ of $\{1, 2, \dots, n\}$ that maximizes $\sum_{i=1}^{n} c_{i\varphi(i)}$ **OR**

- $$\max \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$
$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, \dots, n$$
$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, \dots, n$$
$$x_{ij} \in \{0, 1\} \quad \text{for } i, j = 1, \dots, n$$
OR

- Given a **weighted bipartite graph** $G = (U, V; E)$ with $|U| = |V| = n$ and c_{ij} = cost of edge $(i, j) \in E$, find a perfect matching of maximum value.

- In spite of its simplicity, in the last sixty years the AP attracted hundreds of researchers, accompanying and sometimes anticipating the development of Combinatorial Optimization.
- Formulated in modern way by a psychologist (!),

PSYCHOMETRIKA—VOL. 15, NO. 3
SEPTEMBER, 1950

THE PROBLEM OF CLASSIFICATION OF PERSONNEL*

ROBERT L. THORNDIKE
TEACHERS COLLEGE, COLUMBIA UNIVERSITY

The personnel classification problem arises in its pure form when all job applicants must be used, being divided among a number of job categories. The use of tests for classification involves problems of two types: (1) problems concerning the design, choice, and weighting of tests into a battery, and (2) problems of establishing the optimum administrative procedure of using test results for assignment. A consideration of the first problem emphasizes the desirability of using simple, factorially pure tests which may be expected to have a wide range of validities for different job categories. In the use of test results for assignment, an initial problem is that of expressing predictions of success in different jobs in comparable score units. These units should take account of predictor validity and of job importance. Procedures are described for handling assignment either in terms of daily quotas or in terms of a stable predicted yield.

The past decade, and particularly the war years, have witnessed a great concern about the classification of personnel and a vast expenditure of effort presumably directed towards this end. In all branches of the military establishment were found "general classification" tests or test batteries planned to serve a classification function. Since the war the number of published test batteries designed for differential prediction has rapidly multiplied. It seems timely, therefore, to look into the problem of the classification of personnel to see what the concept means, what issues it raises with respect to the theory of measurement, and what problems it presents with respect to the practical operation of a testing program.

It must be indicated that much of the present discussion represents an examination of concepts, a raising of questions, and an offering of intuitive suggestions, rather than a presentation of mathematically established answers. The defining of questions represents a first step in answering them. It is hoped that clarification of the problems and issues in the following pages may stimulate others to solve them.

Personnel classification, as the term is used here, is best de-

*Address of the President of the Division on Evaluation and Measurement of the American Psychological Association, delivered at Denver, Colorado, September 9, 1949.

215

R.L. Thorndike:

"**Given**: A set of N vacancies to be filled, and N individuals to be used in filling them, **Required**: To assign the individuals to the jobs in such a way that the average success of all the individuals in all the jobs to which they are assigned will be a maximum."

There are, as has been indicated, a finite number of permutations in the assignment of men to jobs. When the classification problem as formulated above was presented to a mathematician, he pointed to this fact and said that from the point of view of the mathematician there was no problem. Since the number of permutations was finite, one had only to try them all and choose the best. He dismissed the problem at that point. This is rather cold comfort to the psychologist, however, when one considers that only ten men and ten jobs mean over three and a half million permutations. Trying out all the permutations may be a mathematical solution to the problem, it is not a practical solution.

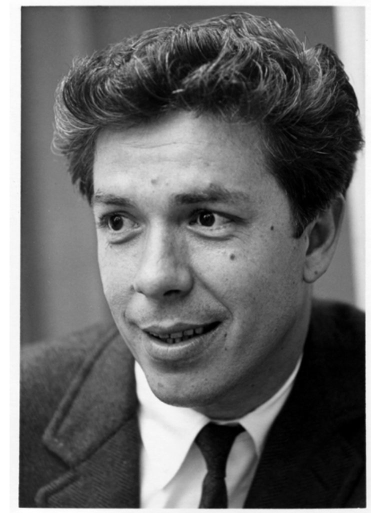
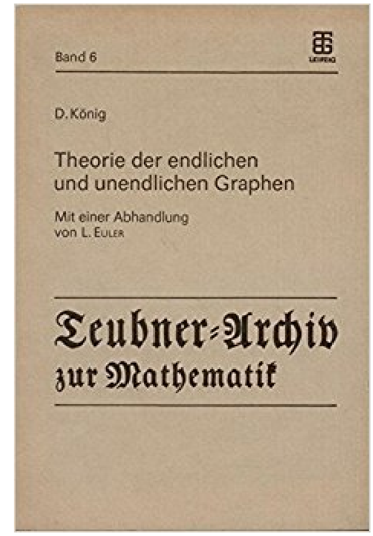
- Complexity will come
  15 years later! (Edmonds, 1965)

- Today, the fastest supercomputer on earth (93 Petaflops) cannot solve a 25×25 AP through enumeration in less than one century.

The Hungarian algorithm

- **H.W. Kuhn** was attracted to the problem in 1953, when C.B. Tompkins, a pioneer in computing, was trying to program a SWAC (the fastest computer in the world) to solve small-size APs.



- Kuhn was reading *Theorie der Endlichen und Unendlichen Graphen* (1936), by Hungarian mathematician **Dénes König**, the first book ever written on Graph Theory, and encountered his augmenting path algorithm for the **maximum cardinality matching** on (non weighted) bipartite graphs.

Denes Kőnig (up to 1944)

- Son of mathematician Gyula Kőnig, he was born in Budapest in 1884 to a family of Jewish origin, but was baptized as a Christian.

- He was a boy prodigy, publishing his first scientific paper at the age of fifteen, while in a Budapest Gymnasium.
- He studied in Budapest and Göttingen (with H. Minkowski), receiving his doctorate in 1907.
- He returned to Budapest, where he worked at the Polytechnic University, becoming Professor in the early Thirties. His lectures were attended, among others, by Egerváry, Erdős, Turán, Gallai.
- During WW2 he worked to help Jewish mathematicians.

Kőnig's matching theorem (1916, 1931)

- **Problem:** Given a bipartite graph $G = (U, V; E)$ (unweighted), find a matching of maximum cardinality.
- **Definition: Vertex cover** = subset C of $U \cup V$ such that every edge is incident with at least one vertex of C.
- **Theorem: The maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover**. **Duality!**
- $\Longleftrightarrow$ matrix property by **Frobenius (1912, 1917)**. However,
- Kőnig gave a beautiful **constructive proof:**
  - given any matching M, there is an **alternating path algorithm** to produce a new matching with cardinality increased by 1;
  - if it fails then $\exists$ a vertex cover having the same cardinality as M.
- The alternating path algorithm is the ancestor of many modern algorithms, including Ford-Fulkerson's max-flow min-cut.

Kőnig (1944)

- From 1920 to 1944 Miklós Horthy led Hungary with a national conservative government. He banned the Hungarian Communist Party as well as the fascist Arrow Cross Party.



- In the late Thirties, Horthy's he was forced to make an alliance with Germany and Italy against Soviet Russia.
- He was however reluctant to contribute to the war effort and he always refused to hand over Hungarian Jews to German authorities.

- In 1944 Horty attempted to strike a secret deal with the Allies.

  The Germans invaded and took control of the country.

  Horthy was removed from power and a puppet government led by the Arrow Cross Party was established.

  The Nazis started deporting Jews (about 250,000) to Auschwitz.

- Kőnig was of Jewish origin, but baptized as a Christian. However,
- in October 1944, fearing to be ordered to move to the ghetto, Kőnig committed suicide.

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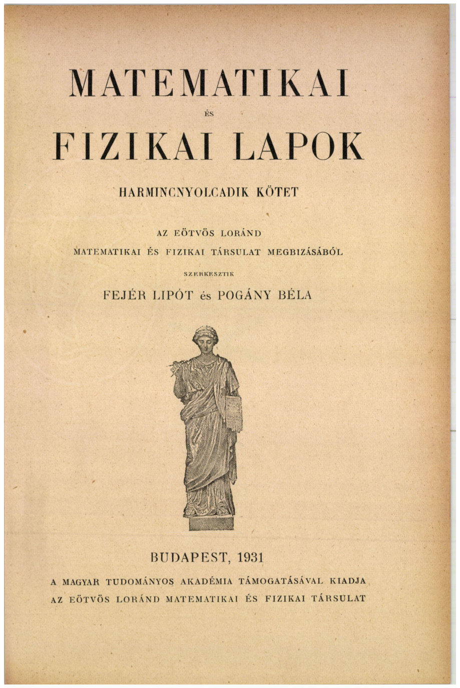
Back to the Fifties, with Kuhn reading Kőnig's book

A footnote in the Kőnig book pointed to a 1931 paper by **Jenö Egerváry**, published in Hungarian on *Matematikai és Fizikai Lapok*.



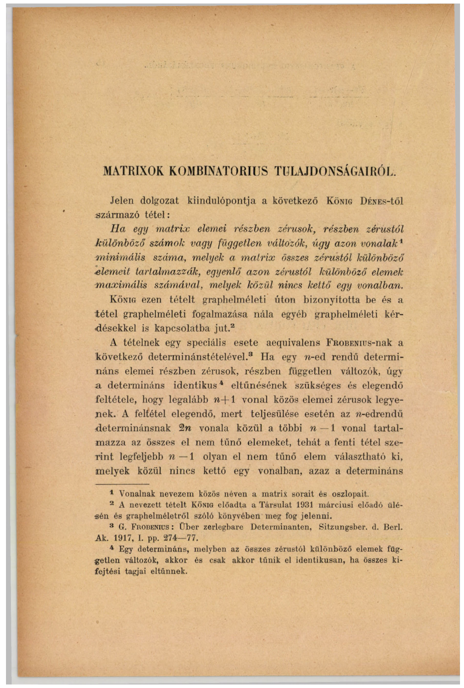
MATEMATIKAI
ÉS
FIZIKAI LAPOK

HARMINCNYOLCADIK KÖTET

AZ EÖTVÖS LORÁND
MATEMATIKAI ÉS FIZIKAI TÁRSULAT MEGBIZÁSÁBÓL
SZERKESZTIK
FEJÉR LIPÓT és POGÁNY BÉLA

BUDAPEST, 1931

A MAGYAR TUDOMÁNYOS AKADÉMIA TÁMOGATÁSÁVAL KIADJA
AZ EÖTVÖS LORÁND MATEMATIKAI ÉS FIZIKAI TÁRSULAT



MATRIXOK KOMBINATORIUS TULAJDONSÁGAIRÓL.

Jelen dolgozat kiindulópontja a következő KÖNIG DÉNES-től származó tétel:

Ha egy matrix elemei részben zérusok, részben zérustól különböző számok vagy független változók, úgy azon vonalak[1] minimális száma, melyek a matrix összes zérustól különböző elemeit tartalmazzák, egyenlő azon zérustól különböző elemek maximális számával, melyek közül nincs kettő egy vonalban.

KÖNIG ezen tételt graphelméleti úton bizonyította be és a tétel graphelméleti fogalmazása nála egyéb graphelméleti kérdésekkel is kapcsolatba jut.[2]

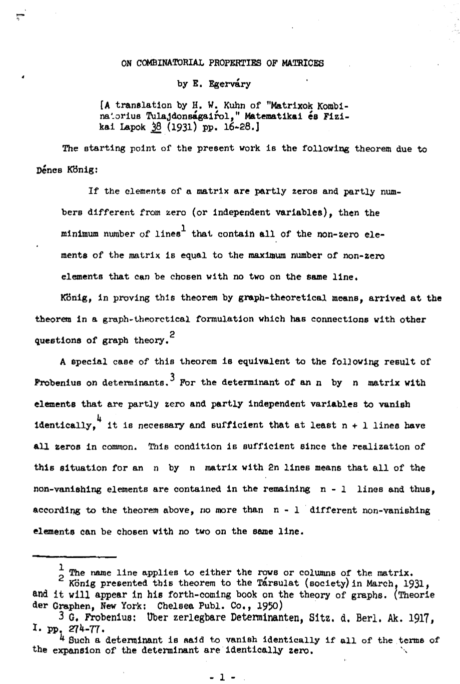
A tételnek egy speciális esete aequivalens FROBENIUS-nak a következő determinánstételével.[3] Ha egy n-ed rendű determináns elemei részben zérusok, részben független változók, úgy a determináns identikus[4] eltűnésének szükséges és elegendő feltétele, hogy legalább $n+1$ vonal közös elemei zérusok legyenek. A feltétel elegendő, mert teljesülése esetén az n-edrendű determinánsnak $2n$ vonala közül a többi $n-1$ vonal tartalmazza az összes el nem tűnő elemeket, tehát a fenti tétel szerint legfeljebb $n-1$ olyan el nem tűnő elem választható ki, melyek közül nincs kettő egy vonalban, azaz a determináns

[1] Vonalnak nevezem közös néven a matrix sorait és oszlopait.

[2] A nevezett tételt KÖNIG előadta a Társulat 1931 márciusi előadó ülésén és graphelméletről szóló könyvében meg fog jelenni.

[3] G. FROBENIUS: Über zerlegbare Determinanten, Sitzungsber. d. Berl. Ak. 1917, I. pp. 274—77.

[4] Egy determináns, melyben az összes zérustól különböző elemek független változók, akkor és csak akkor tűnik el identikusan, ha összes kifejtési tagjai eltűnnek.



ON COMBINATORIAL PROPERTIES OF MATRICES

by E. Egerváry

[A translation by H. W. Kuhn of "Matrixok Kombinatorius Tulajdonságairól," Matematikai és Fizikai Lapok 38 (1931) pp. 16-28.]

The starting point of the present work is the following theorem due to Dénes König:

If the elements of a matrix are partly zeros and partly numbers different from zero (or independent variables), then the minimum number of lines[1] that contain all of the non-zero elements of the matrix is equal to the maximum number of non-zero elements that can be chosen with no two on the same line.

König, in proving this theorem by graph-theoretical means, arrived at the theorem in a graph-theoretical formulation which has connections with other questions of graph theory.[2]

A special case of this theorem is equivalent to the following result of Frobenius on determinants.[3] For the determinant of an n by n matrix with elements that are partly zero and partly independent variables to vanish identically,[4] it is necessary and sufficient that at least $n + 1$ lines have all zeros in common. This condition is sufficient since the realization of this situation for an n by n matrix with $2n$ lines means that all of the non-vanishing elements are contained in the remaining $n - 1$ lines and thus, according to the theorem above, no more than $n - 1$ different non-vanishing elements can be chosen with no two on the same line.

[1] The name line applies to either the rows or columns of the matrix.

[2] König presented this theorem to the Társulat (society) in March, 1931, and it will appear in his forth-coming book on the theory of graphs. (Theorie der Graphen, New York: Chelsea Publ. Co., 1950)

[3] G. Frobenius: Über zerlegbare Determinanten, Sitz. d. Berl. Ak. 1917, I. pp. 274-77.

[4] Such a determinant is said to vanish identically if all of the terms of the expansion of the determinant are identically zero.

- 1 -

Kuhn translated Egerváry's paper, and published the translation as a Research Report of the George Washington University: *On Combinatorial Properties of Matrices*. It contains **2 theorems**.

Jenő Egerváry (up to 1955)

- He was born in Debrecen (Hungary) in 1891. In 1914, he received his doctorate under the supervision of Lipót Fejér (thesis advisor of John von Neumann, Paul Erdős, George Pólya, and Pál Turán).

- He had a wide scientific production ranging from the theory of algebraic equations to geometry, from differential equations to matrix theory, mostly with an applied flavor.
- In 1931, shortly after Kőnig lectured on his result for the matching problem, he wrote the paper mentioned in Kőnig's book.
- In 1941 he became full professor at the T.U. of Budapest.
- in 1955 he became head of the Department of Mathematics.

First Egerváry's theorem

- Remind the **Duality of the Assignment problem** (**Fifties**):

$$
\begin{aligned}
(P)\ \max \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1 & (i = 1, 2, \dots, n), \\
& \sum_{i=1}^{n} x_{ij} = 1 & (j = 1, 2, \dots, n), \\
& x_{ij} \in \{0, 1\} & (i, j = 1, 2, \dots, n). \\
(D)\ \min \quad & \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j & \\
\text{s.t.} \quad & u_i + v_j \geq c_{ij} & (i, j = 1, 2, \dots, n).
\end{aligned}
$$

Strong Duality: $\min \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \;=\; \max \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$

First Egerváry's theorem

- **Covering system** = set of *lines* (rows and columns) that contain the ith row of an $n \times n$ matrix C with multiplicity λ_i and the jth column with multiplicity μ_j, and satisfy

$$\lambda_i + \mu_j \geq c_{ij} \quad (i, j = 1, \ldots, n). \tag{1}$$

- **Minimal covering system** = covering system of minimum value

$$\sum_{k=1}^{n} (\lambda_k + \mu_k) \tag{2}$$

- **Minimizing (2) subject to (1) is the dual of the AP.**

- **First Egerváry's Theorem: If (c_{ij}) is an $n \times n$ matrix of non-negative integers then, subject to (1), we have**

$$\min \sum_{k=1}^{n} (\lambda_k + \mu_k) = \max_{\varphi} \sum_{i=1}^{n} c_{i\varphi(i)}.$$

Jenő Egerváry (1956–1958)

- In October 1956 a student demonstration started nationwide revolution against the government of the Hungarian People's Republic and its Soviet-imposed policies.

- The revolt spread across Hungary and the Communist government quickly collapsed.
  A new democratic government was established.

- On November 4th, a large Soviet force invaded Budapest. The Hungarian resistance continued until 10 November. Over 2,500 Hungarians and 700 Soviet troops were killed. 200,000 Hungarians left the country as refugees.
- By January 1957, the new Soviet-installed government had suppressed all public opposition.

- Egerváry was head of the Department of Mathematics, but in the subsequent repression period he was forced to retire under specious financial pretexts.
  In 1958, fearing to be imprisoned, Egerváry committed suicide.

Back to Kuhn

- Egerváry proved his theorem by giving an algorithm that iteratively adjusts the current (feasible, non optimal) λ_i and μ_j values so that
  (i) they remain feasible;
  (ii) their sum decreases
  until the optimal solution is found.

- Using Egerváry's method and Kőnig's maximum matching algorithm, in the fall of 1953 Kuhn solved several 12×12 assignment problems **by hand**.

- Each of these examples took under two hours to solve.
  This must have been one of the last times when pencil and paper could beat the largest and fastest electronic computer in the world. (H.W. Kuhn, EURO XXIV, Lisbon, July 2010)

- The algorithm was christened the **Hungarian algorithm** in honor of these two mathematicians and published in two famous papers on *Naval Research Logistics Quarterly*:

  "The Hungarian method for the assignment problem" (1955)

  "Variants of the Hungarian method for the assignment problem" (1956)

  Reprinted in 2005: "The Hungarian method for the assignment problem"

  Over **7000 citations** on Google Scholar.

- **But this is not the end of the story.**
  A recent historical discovery: A posthumous paper written, prior to **1851 (!!!)** by one of the greatest mathematicians of all time.

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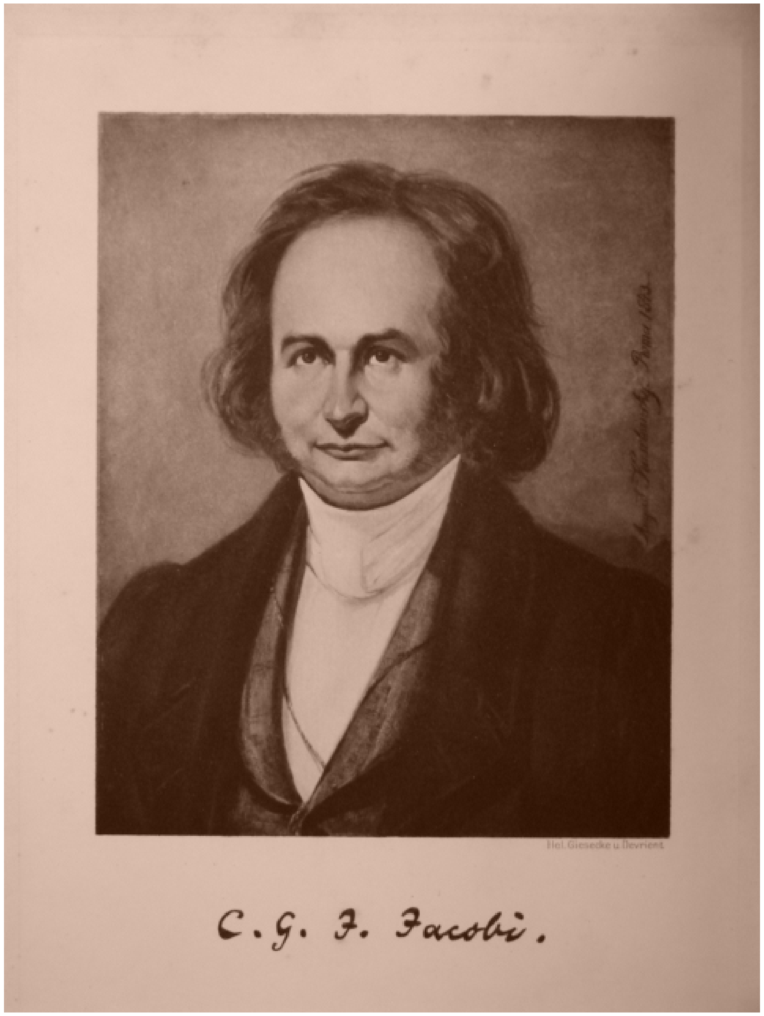


DE INVESTIGANDO ORDINE SYSTEMATIS
AEQUATIONUM DIFFERENTIALIUM
VULGARIUM CUJUSCUNQUE

AUCTORE

C. G. J. JACOBI,
PROF. ORD. MATH. REGIOM.

Borchardt Journal für die reine und angewandte Mathematik, Bd. 64 p. 297—320.

About the research of the order of a system of arbitrary ordinary differential equations (posthumous manuscript).

2.

De solutione problematis inaequalitatum, quo investigatio ordinis systematis aequationum differentialium quarumcunque innititur. Proposito schemate, definitur Canon. Dato Canone quocunque, invenitur simplicissimus.

Antecedentibus investigatio ordinis systematis aequationum differentialium vulgarium revocata est ad sequens problema inaequalitatum etiam per se tractatu dignum:

Problema.

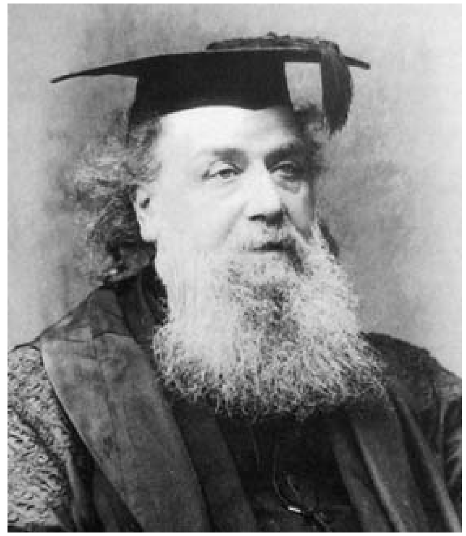
Disponantur nn quantitates $h_k^{(i)}$ *quaecunque in schema Quadrati, ita ut habeantur n series horizontales et n series verticales, quarum quaeque est n terminorum. Ex illis quantitatibus eligantur n transversales, i. e. in seriebus horizontalibus simul atque verticalibus diversis positae, quod fieri potest* $1.2\ldots n$ *modis; ex omnibus illis modis quaerendus est is, qui summam n numerorum electorum suppeditet maximam.*

Dispositis quantitatibus $h_k^{(i)}$ in figuram quadraticam

$$\begin{matrix} h_1' & h_2' & \ldots & h_n' \\ h_1'' & h_2'' & \ldots & h_n'' \\ \cdot & \cdot & \cdot & \cdot \\ h_1^{(n)} & h_2^{(n)} & \ldots & h_n^{(n)}, \end{matrix}$$

earum systema appellabo *schema propositum;* omne schema, inde ortum addendo singulis ejusdem seriei horizontalis terminis eandem quantitatem, appellabo *schema derivatum.* Sit

Note: Jacobi did not even have proper terminology for a “matrix”!
The term was coined in same years by James J. Sylvester:



This homaloidal law has not been stated in the above commentary in its form of greatest generality. For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding to which may be termed determinants of the pth order. We have, then, the following proposition. The number of uncoevanescent determinants constituting a system of the pth order derived from a given matrix, n terms broad and m terms deep, may equal, but can never exceed the number

$$(n-p+1)(m-p+1).$$

matrix (n.) late 14c., "uterus, womb," from Old French *matrice*.

Canon derivatus I.

	I	II	III	IV	V	VI	VII	*t*
I	11*	11	8	19	18	10	5	4
II	10	15*	14	13	18	21	17	7
III	8	13	17*	18	17	25	12	2
IV	4	11	14	25*	20	21	27	0
V	9	6	12	14	27*	22	34	4
VI	6	13	8	14	11	25*	22	5
VII	11	12	8	22	24	21	40*	0

Canon derivatus II.

	I	II	III	IV	V	VI	VII	*t*
I	11*	11	8	19	18	10	5	4
II	8	13*	12	11	16	19	15	5
III	6	11	15*	16	15	23	10	0
IV	4	11	14	25*	20	21	27	0
V	7	4	10	12	25*	20	32	2
VI	4	11	6	12	9	23*	20	3
VII	11	12	8	22	24	21	40*	0

Canon simplicissimus.

	I	II	III	IV	V	VI	VII	*t*
I	11*	11	8	19	18	10	5	4
II	7	12*	11	10	15	18	14	4
III	6	11	15*	16	15	23	10	0
IV	4	11	14	25*	20	21	27	0
V	6	3	9	11	24*	19	31	1
VI	4	11	6	12	9	23*	20	3
VII	11	12	8	22	24	21	40*	0

E schemate proposito, addendo terminis serierum diversarum respective

The Jacobi method replicates the patterns of the Hungarian Method. **It IS the Hungarian Algorithm!**

1. **The Fifties: Birth of Combinatorial Optimization**
2. **The Assignment Problem and Kőnig's Matching Theorem**
3. **The First Egerváry's Theorem (1931) and Strong Duality**
4. **Jacobi (1851) and the Hungarian method**
5. **Open shop, satellite communications, and the Second Egerváry's Theorem**

A result that has been "discovered" several times:

Telecommunication systems using satellites (TDMA):

- the data, buffered in ground stations, are remitted to the satellite where they are sent back to earth;
- onboard the satellite n transponders connect the sending stations with the receiving stations;
- $n \times n$ matrix C, c_{ij} = time needed to transfer the required data from station i to station j $(i, j = 1, \ldots, n)$.
- Find a transmission schedule such that all data are transmitted in minimum time, i.e., find
- permutation matrices P_k (connections) and transmission times λ_k st

$$\begin{array}{lll} \min & \sum_k \lambda_k & \\ \text{s.t.} & \sum_k \lambda_k P_k \geq C & \text{elementwise} \\ & \lambda_k \geq 0 & \end{array}$$

Preemptive Open Shop Scheduling:

- we are given n machines and n jobs, and each job j must be processed on every machine i (in any order) for c_{ij} time units;
- each machine can process at most one job at a time and no job can be processed simultaneously on two machines;
- each processing can be interrupted at any time and resumed later.
- Find a schedule such that the completion time of the latest job is as small as possible, i.e.,
- find permutation matrices P_k and processing times λ_k such that

$$\begin{array}{lll} \min & \sum_k \lambda_k & \\ \text{s.t.} & \sum_k \lambda_k P_k \geq C & \text{elementwise} \\ & \lambda_k \geq 0 & \end{array}$$

Polynomial-time algorithm by **Gonzalez and Sahni** (*J. ACM*, 1976).
Algorithm by **Inukai** (*IEEE Trans. Comm.*, 1979). **Identical:**

0. $c^* := \max\left(\text{(max row sum)}, \text{(max column sum)} \right)$,
a lower bound for $\min \sum_k \lambda_k$.

1. Add dummy values to C st all row and column sums $= c^*$;

2. Iteratively subtract permutation matrices P_k, with $\lambda_k =$ smallest positive cost corresponding to an element of P_k, until $C = \underline{\mathbf{0}}$.

- **Comment:** The modified matrix is the integer version of a **doubly stochastic matrix** (a square non-negative matrix with all row and column sums equal to 1).
- Famous result by **Garrett Birkhoff** on doubly stochastic matrices, published (**in Spanish**) by Garrett Birkhoff in **1946** and for which John von Neumann gave in **1953** an elegant proof.

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SERIE A

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Es evidente que cada matriz A que satisface (1) satisface también

$$(1') \qquad \sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1, \text{ para todo } i, j = 1, \ldots, n.$$

Estas matrices son interesantes para la probabilidad,[1] y los cuadrados mágicos son múltiplos escalares de estas matrices.

Teorema. Si una matriz $n \times n$ A satisface (1′), entonces es una media aritmética de permutaciones.

Theorem: Every doubly stochastic matrix is a convex combination of permutation matrices.

Second Egerváry's theorem

- Given a **non-negative integer** $n \times n$ **matrix** C, consider the $n!$ distinct **permutation matrices** $P^k = (p_{ij}^k)$:

- a system of permutation matrices which contains the kth matrix, P^k, with **multiplicity** λ_k $(\lambda_k \geq 0 \ \forall \ k)$ is called a **diagonal covering system** for C if

$$\sum_{k=1}^{n!} \lambda_k p_{ij}^k \geq c_{ij} \quad (i, j = 1, 2, \ldots, n).$$

- **Problem:**

  find a diagonal covering system of minimum value $\displaystyle\sum_{k=1}^{n!} \lambda_k$.

- **Theorem:** $\min \sum_{k=1}^{n!} \lambda_k = \max(\max_i \sum_{j=1}^{n} c_{ij}, \max_j \sum_{i=1}^{n} c_{ij})(= c^*)$.

Proof:

1. Define a **majorant** of C, i.e., a matrix C^* such that

$$c^*_{ij} \geq c_{ij} \quad \text{and} \quad \sum_{i=1}^{n} c^*_{ij} = \sum_{j=1}^{n} c^*_{ij} = c^* \quad (i, j = 1, 2, \ldots, n).$$

2. Iteratively subtract from C^* permutation matrices P such that, for at least one (i, j), $c^*_{ij} > 0$ if $p_{ij} = 1$, until C^* becomes a zero matrix.

- **The algorithms of the Seventies, in 1931!**

Egerváry's bust erected in the University gardens (1992, 2006).

Thank you for your attention

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