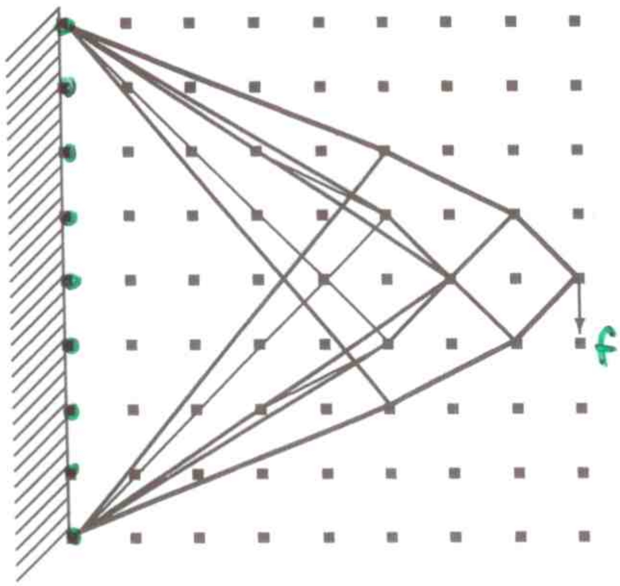
Optimal Design of Engineering Structures

TRUSS TOPOLOGY DESIGN

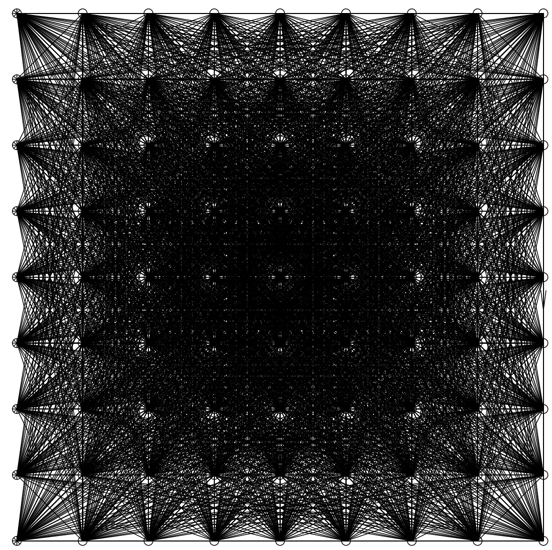
Engineering Formulation: Given total volume of a truss (construction comprised of thin elastic bars linked to each other, like electricity mast or Eifel Tower), find the truss which is most rigid with respect to a given set of external loads.

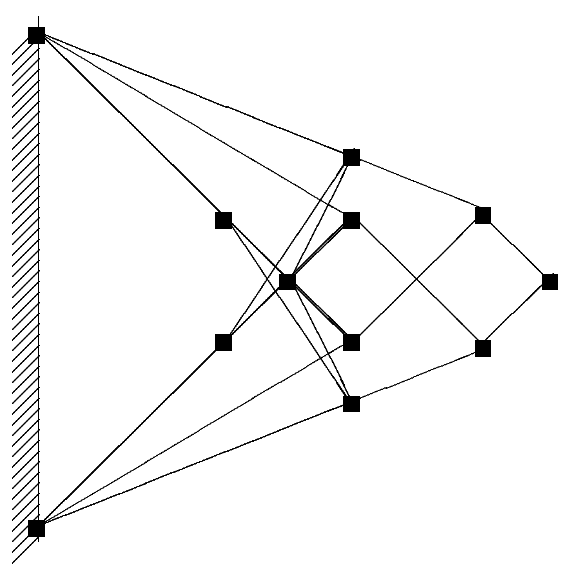


A Truss

Problem Type \ Data/Design Variables	Location of Nodes	Connectivity Pattern	Bar Volumes (Cross Sec. Areas)
Sizing	+	+	?
Topology	+	?	?
Geometry	?	?	?

Truss Topology Design





<u>The simplest TTD problem</u> is

$$\text{Compliance} = \tfrac{1}{2} f^T x \quad \to \quad \min$$

s.t.

$$\begin{array}{rcl} \underbrace{\left[\sum_{i=1}^{m} t_i b_i b_i^T\right]}_{A(t)\succeq 0} x & = & f \\ \sum_{i=1}^{m} t_i & \leq & w \\ t & \geq & 0 \end{array}$$

- **Data:**

— $b_i \in \mathbb{R}^n$**,** n **−** **# of nodal degrees of freedom (for a** $10 \times 10 \times 10$ **ground structure,** $n \approx 3,000$**)**

— m **− # of tentative bars (for** $10 \times 10 \times 10$ **ground structure,** $m \approx 500,000$**)**

- **Design variables:** $t \in \mathbb{R}^m$**,** $x \in \mathbb{R}^n$

IMPORTANCE OF RICH TOPOLOGY

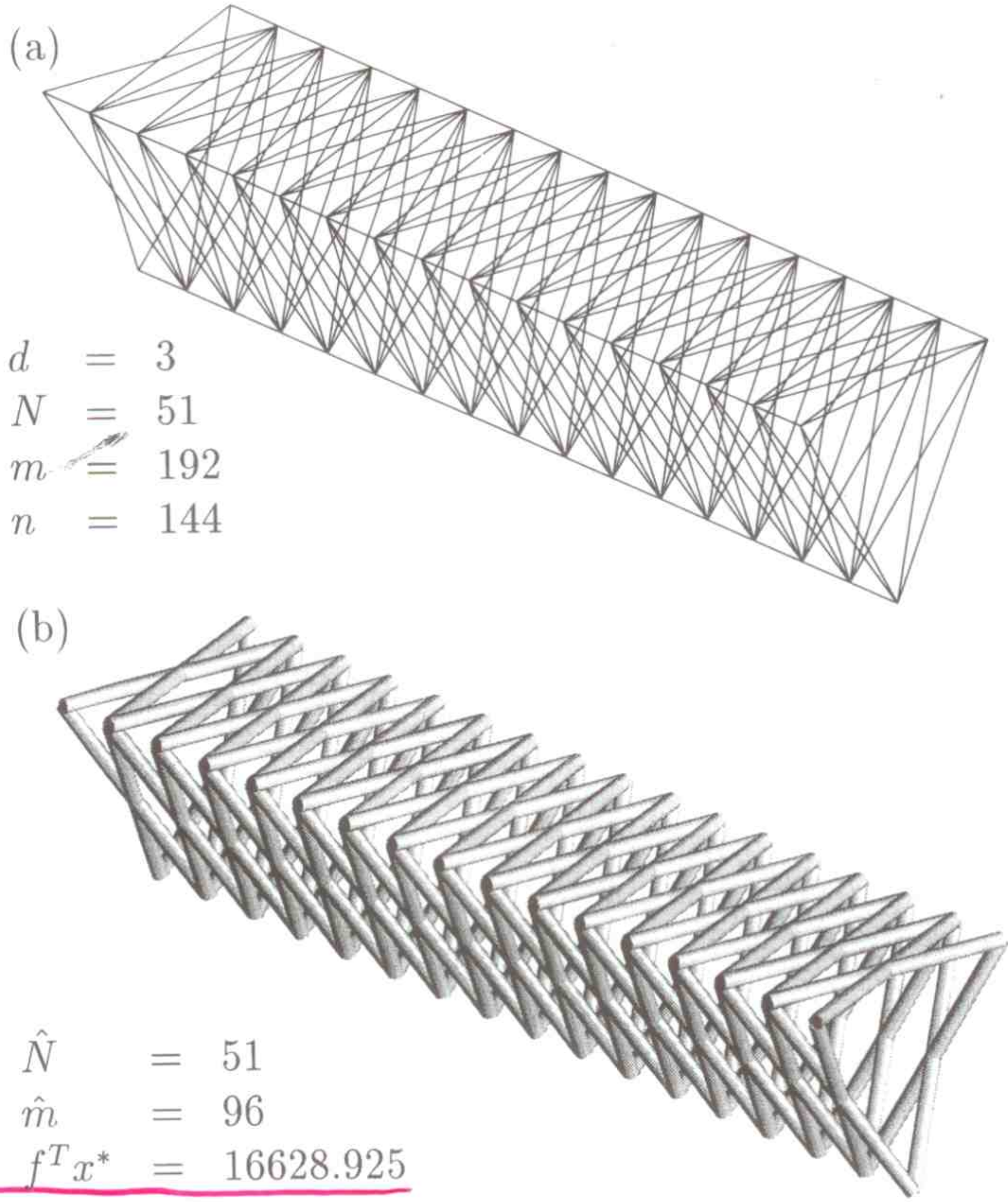
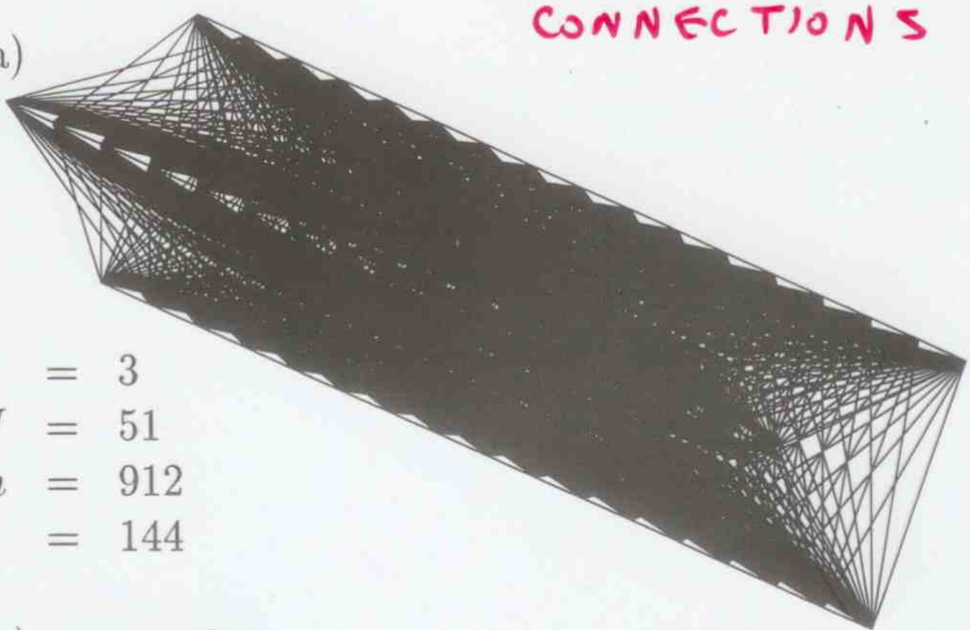


Figure 2: Triangle cantilever arm under tension (only neighbours connected)

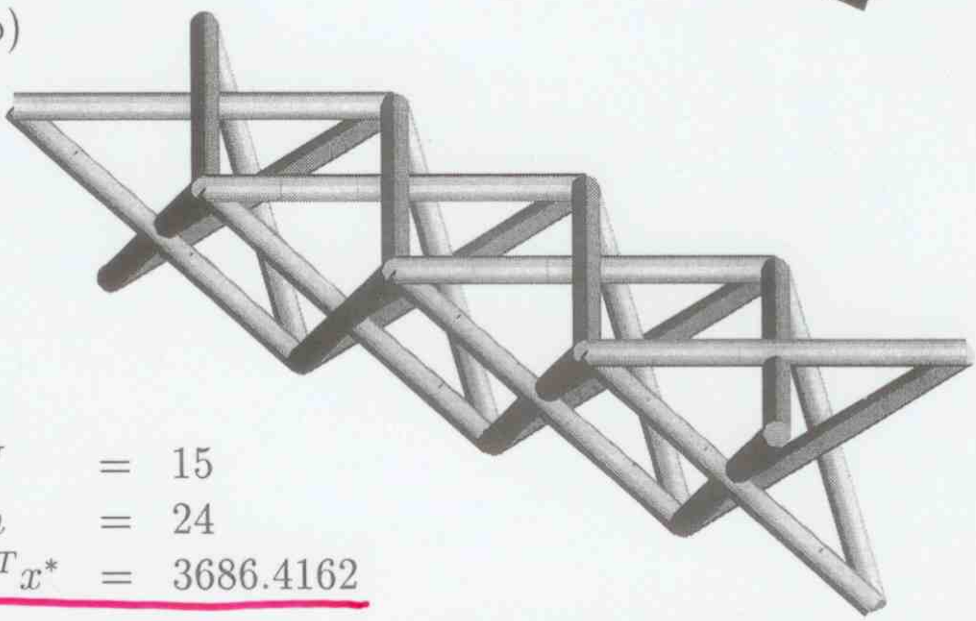
ALL POSSIBLE CONNECTIONS

(a)



$$d = 3$$
$$N = 51$$
$$m = 912$$
$$n = 144$$

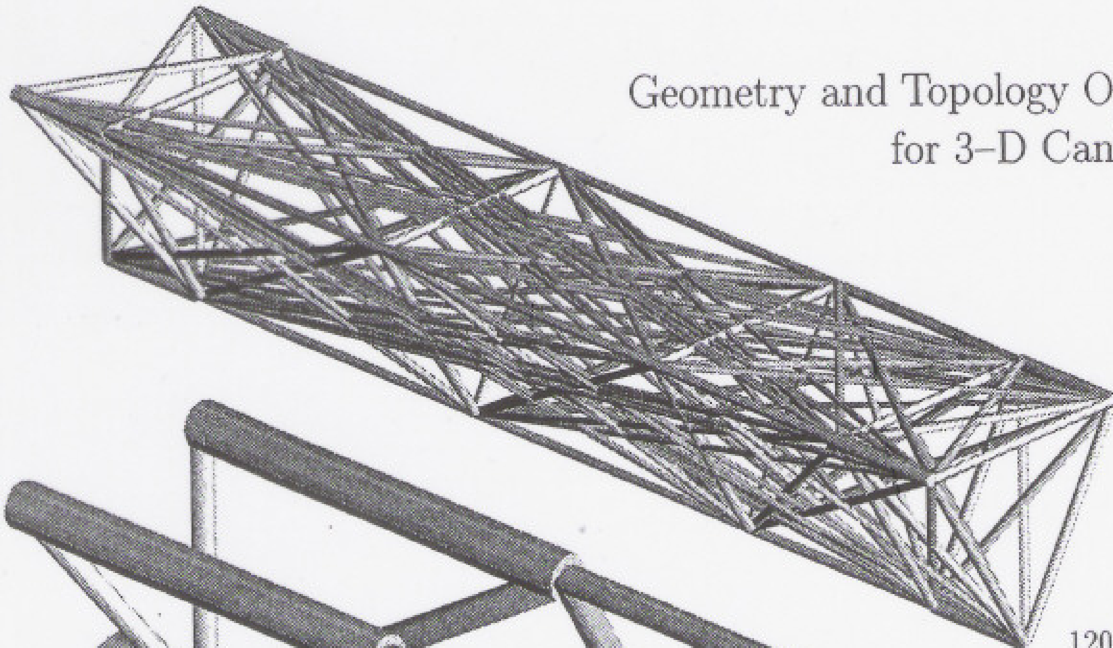
(b)



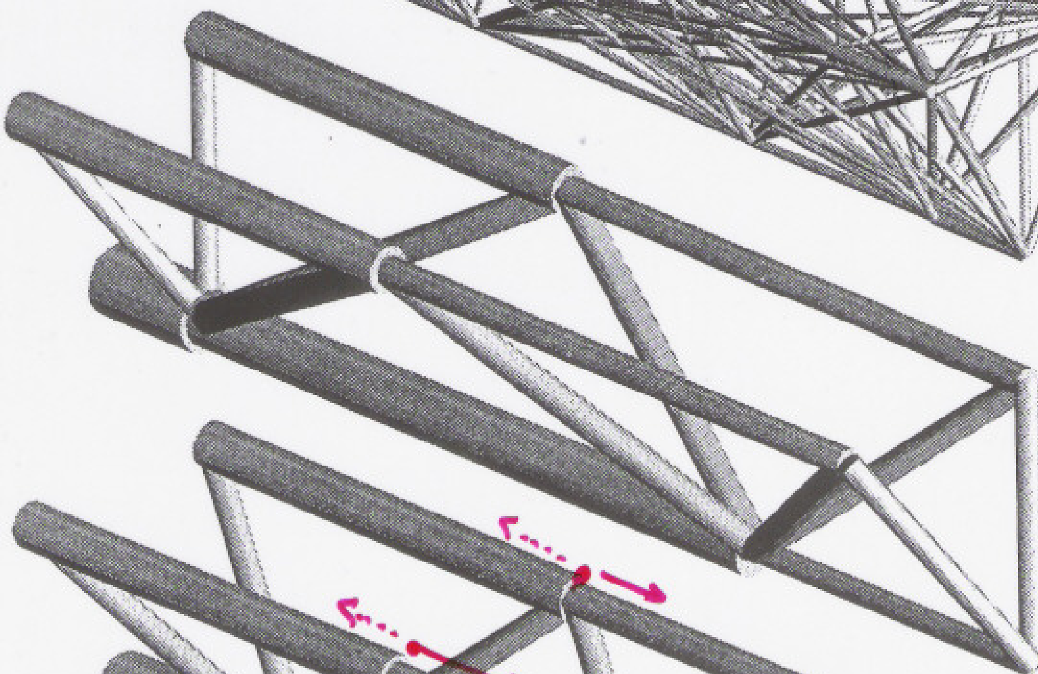
$$\hat{N} = 15$$
$$\hat{m} = 24$$
$$f^T x^* = 3686.4162$$

Figure 3: Triangle cantilever arm under tension (all connections)

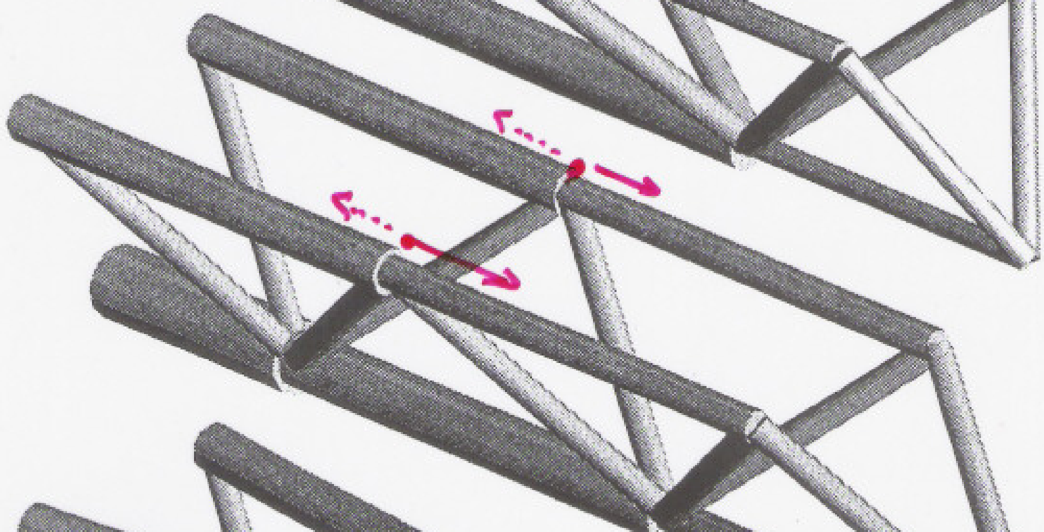
Geometry and Topology Optimization for 3–D Cantilever Arm



120 potential bars



17 bars;
compl. 6.944



x-coord. optimized
17 bars;
compl. 6.563

x- and z-coord. opt.
18 bars;
compl. 6.326

Multi-Load TTD

$$\text{(P)} \qquad \min_{x_j, t} \max_{j=1,\ldots,k} \{f_j^T x_j\}$$

s.t.

$$\sum_{i=1}^m t_i A_i x_j = f_j$$

$$\sum_{i=1}^m t_i = v$$

$$t_i = 0\,, \qquad i = 1, \ldots, m$$

$$x^j \in I\!R^n, \quad j = 1, \ldots, k$$

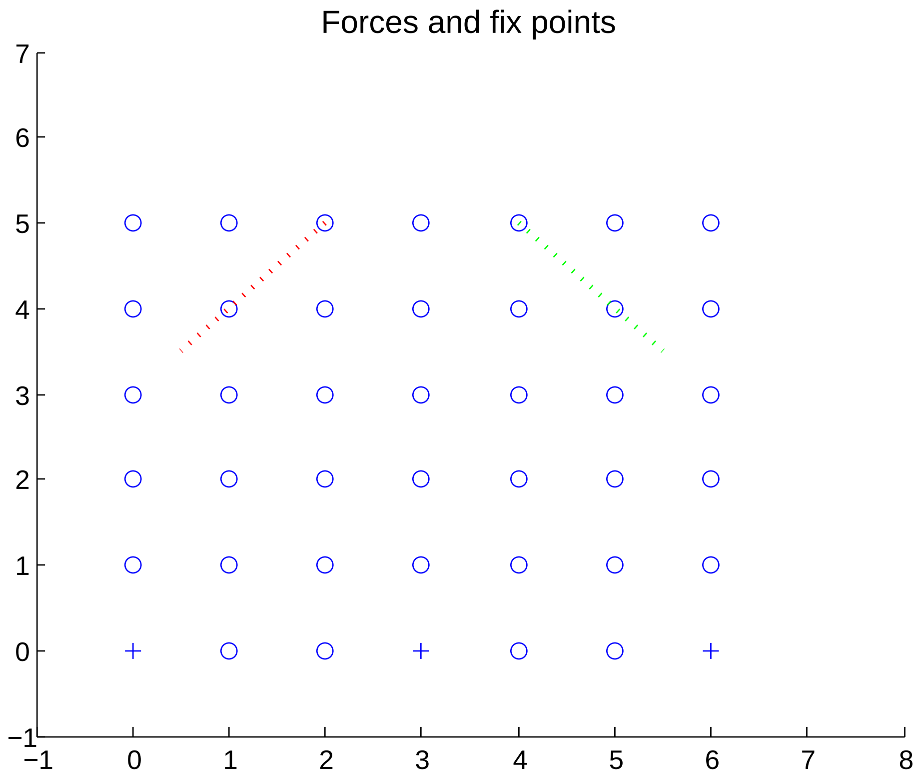
$$\boxed{\begin{array}{l} \displaystyle\min_{\substack{x_j, \ldots, x_k \\ \lambda \in R^k}} \left\{ \sum_{j=1}^k f_j^T x_j + v \cdot \max_{i=1,\ldots,m} \left\{ \sum_{j=1}^K \frac{x_j^T A_i x_j}{\lambda_j} \right\} \right\} \\ \text{s.t.} \\ \displaystyle\qquad \sum_{j=1}^k \lambda_j = 1, \quad \lambda \geq 0\,. \end{array}}$$

CONVEX PROGRAM

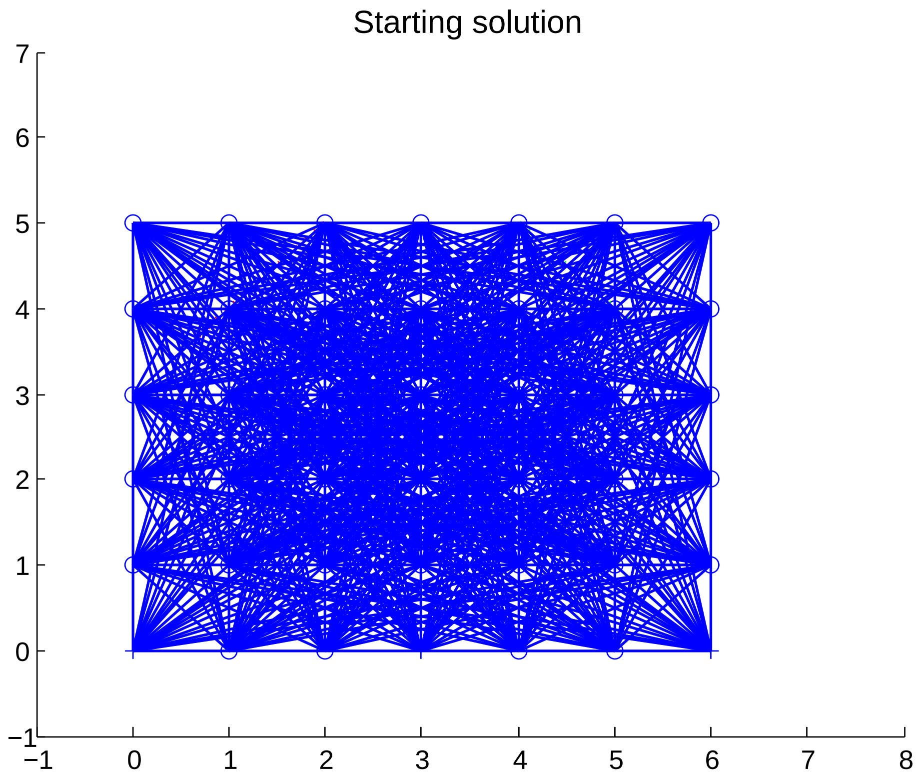
$$\min_{\substack{x_1,\ldots,x_k \\ \lambda\in R^k \\ \tau\in R}} \left\{\sum f_j^T x_j + v\tau\right\}$$

$$\sum \lambda_j = 1, \quad \lambda \geq 0$$

$$\begin{array}{c}\text{dual var.}\\ t_j^*\end{array} \quad \Rightarrow \quad \underbrace{\sum_j \left(\frac{x_j^T A_j x_j}{\lambda_j}\right) \leq \tau}_{\text{a conic quadratic constraint}} \qquad \forall\, i$$

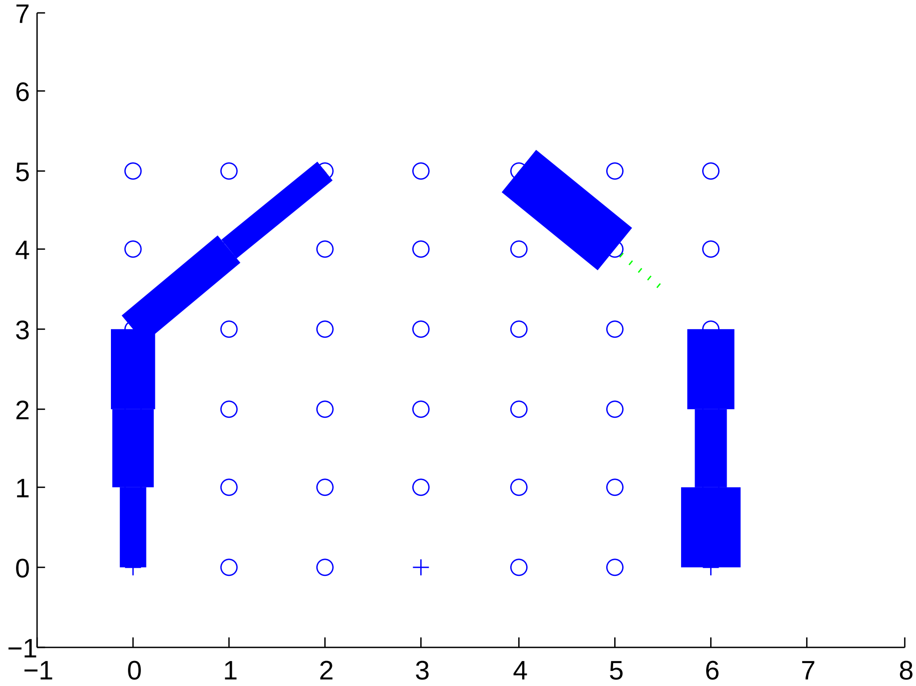


Forces and fix points
7
6
5
4
3
2
1
0
−1
−1
0
1
2
3
4
5
6
7
8

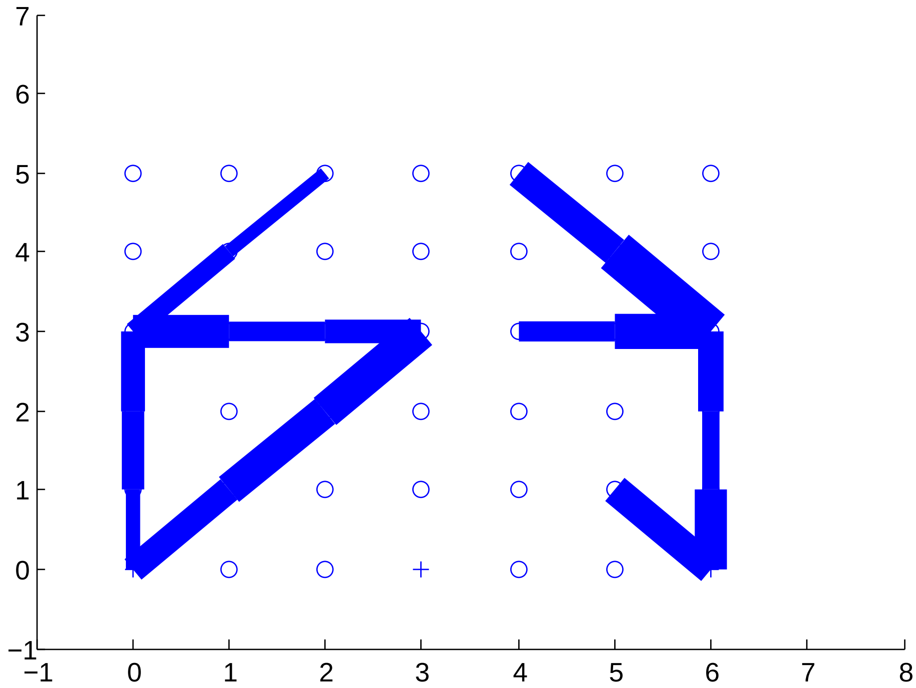


Starting solution
7
6
5
4
3
2
1
0
−1
−1
0
1
2
3
4
5
6
7
8

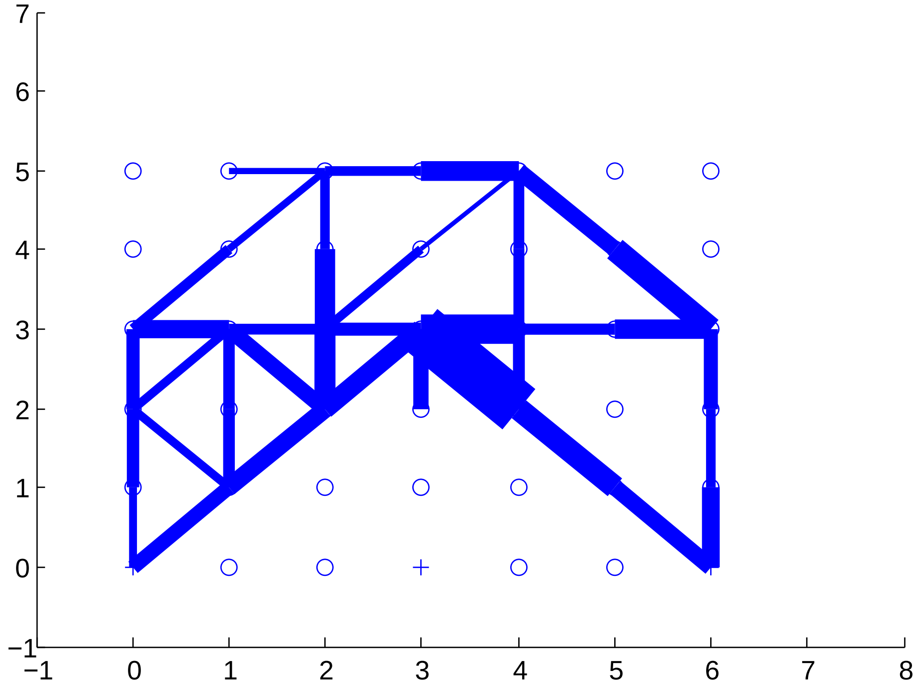
iteration number 10



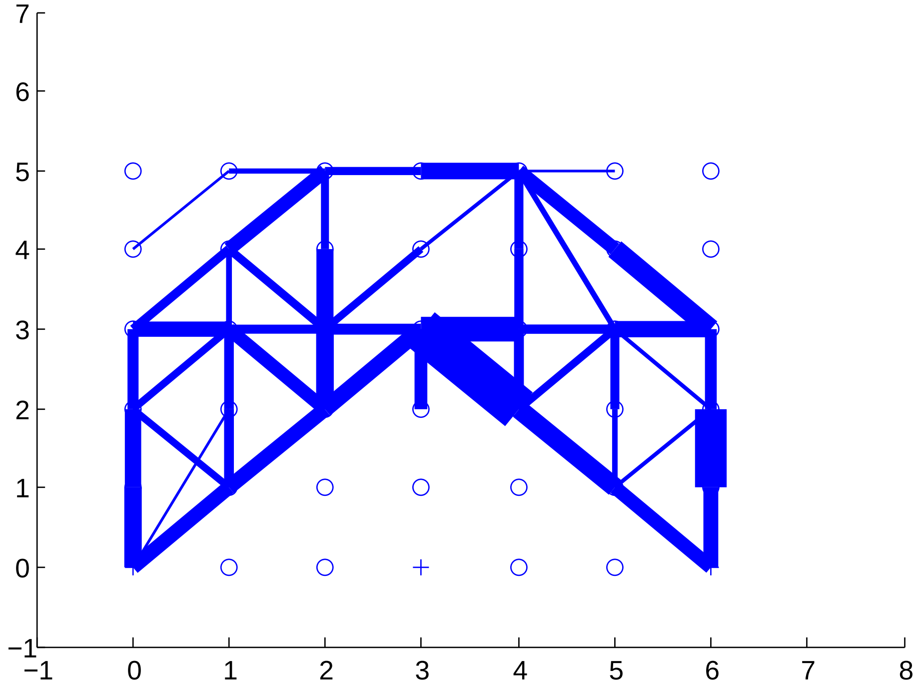
iteration number 20



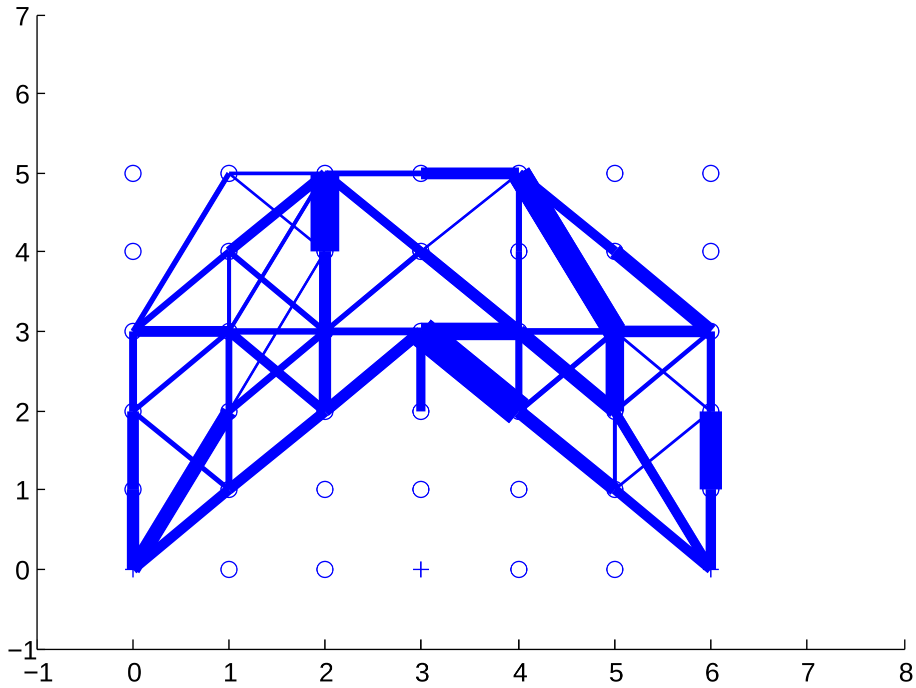
iteration number 40

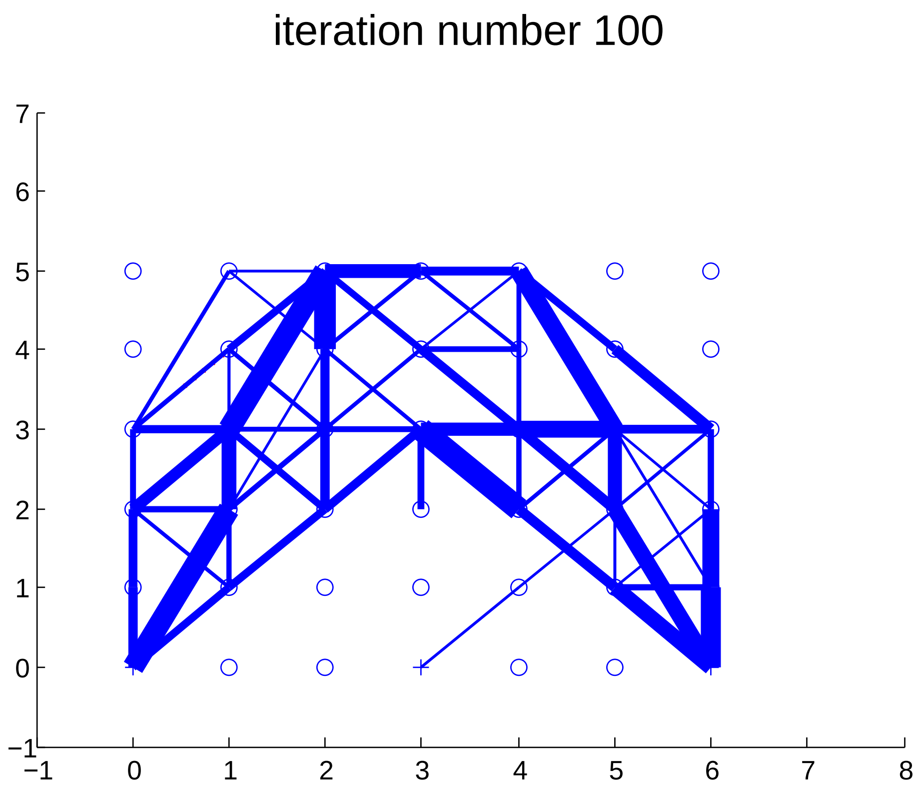


iteration number 60

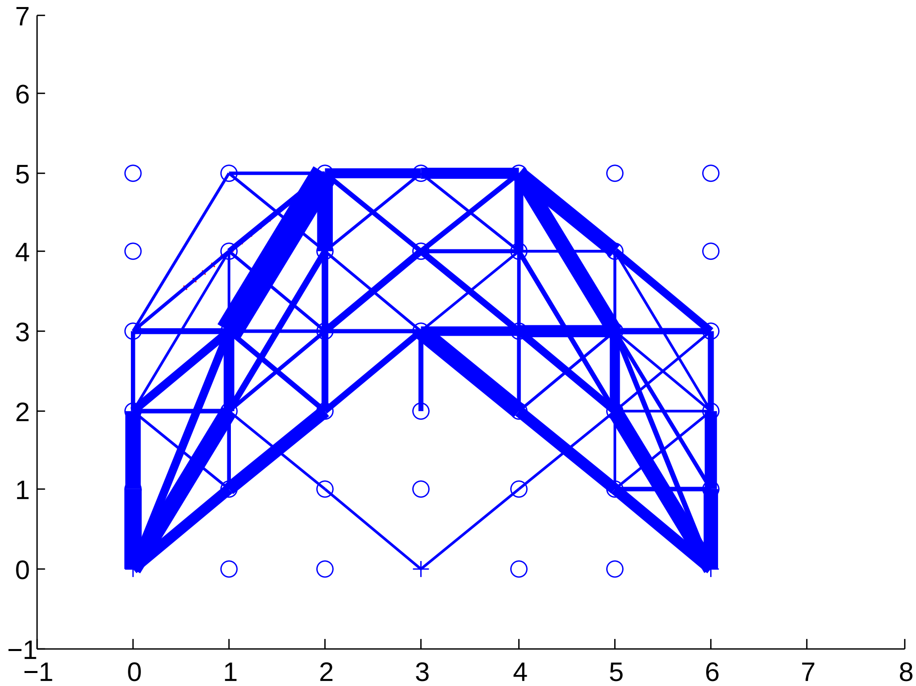


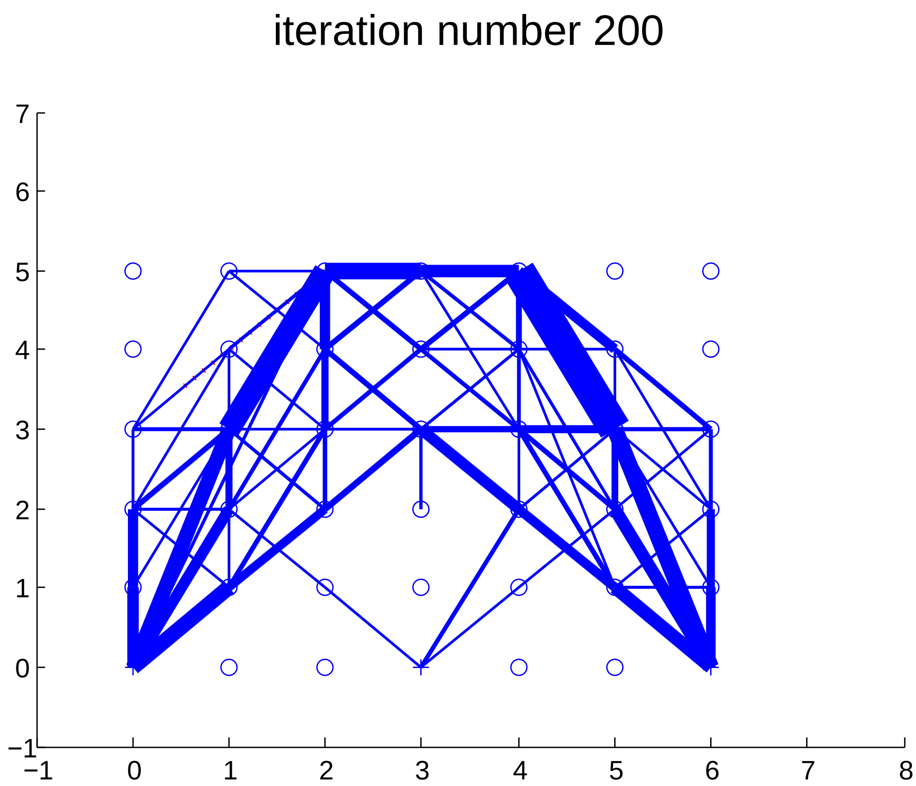
iteration number 80



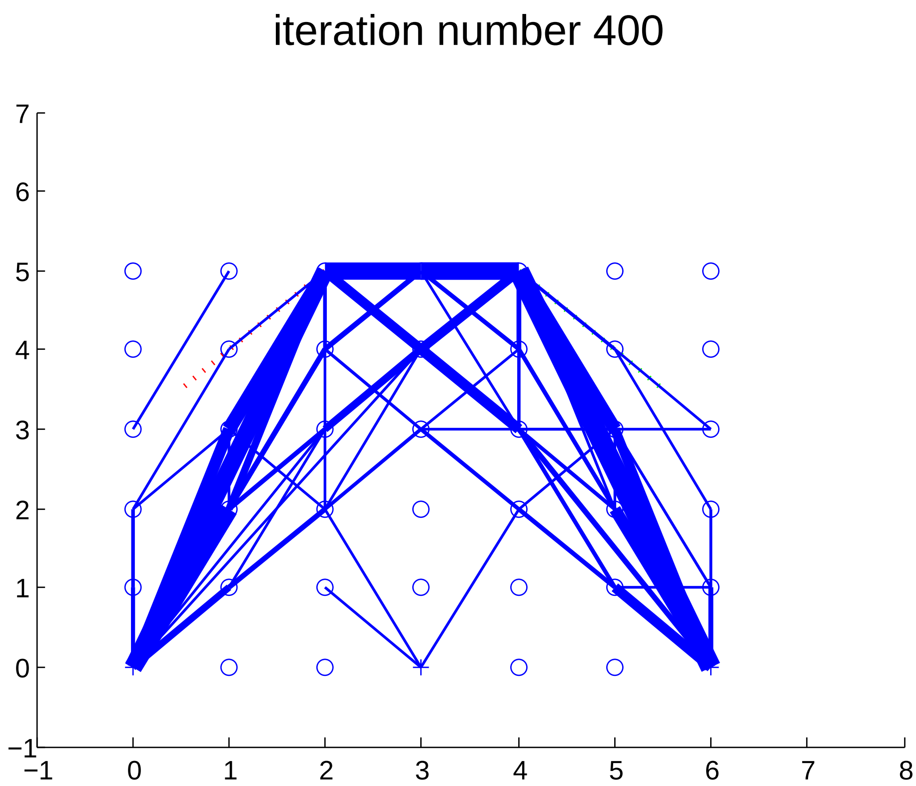

iteration number 100
7
6
5
4
3
2
1
0
−1
−1
0
1
2
3
4
5
6
7
8

iteration number 150

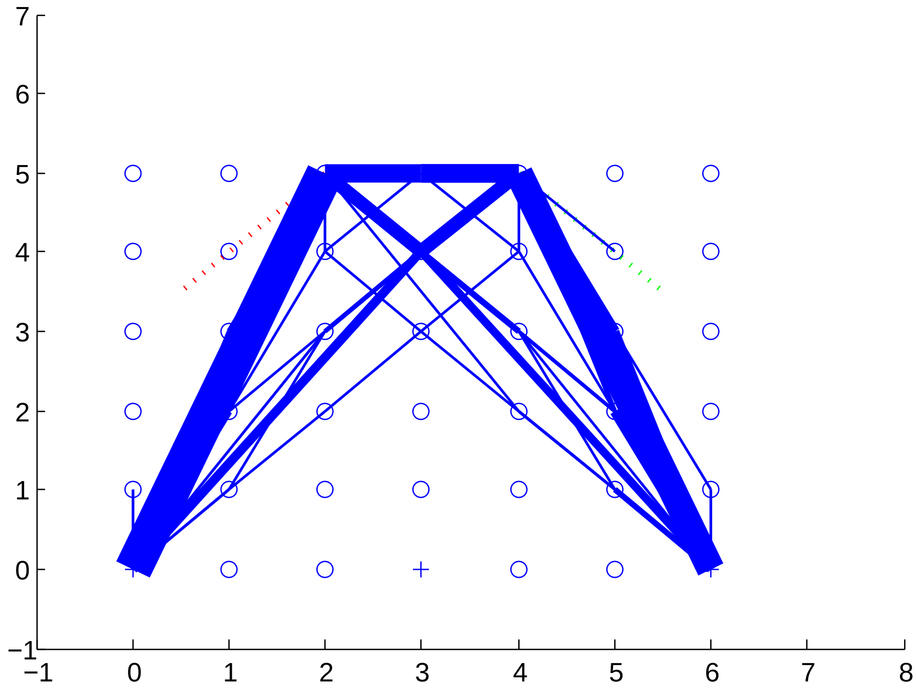




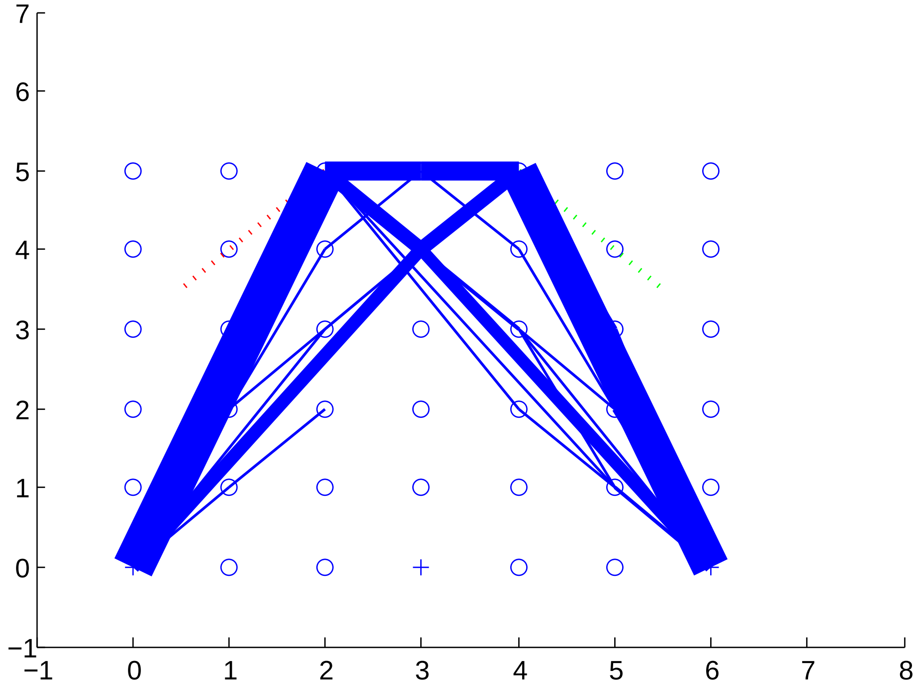
iteration number 200
7
6
5
4
3
2
1
0
−1
−1
0
1
2
3
4
5
6
7
8


iteration number 400
7
6
5
4
3
2
1
0
−1
−1
0
1
2
3
4
5
6
7
8

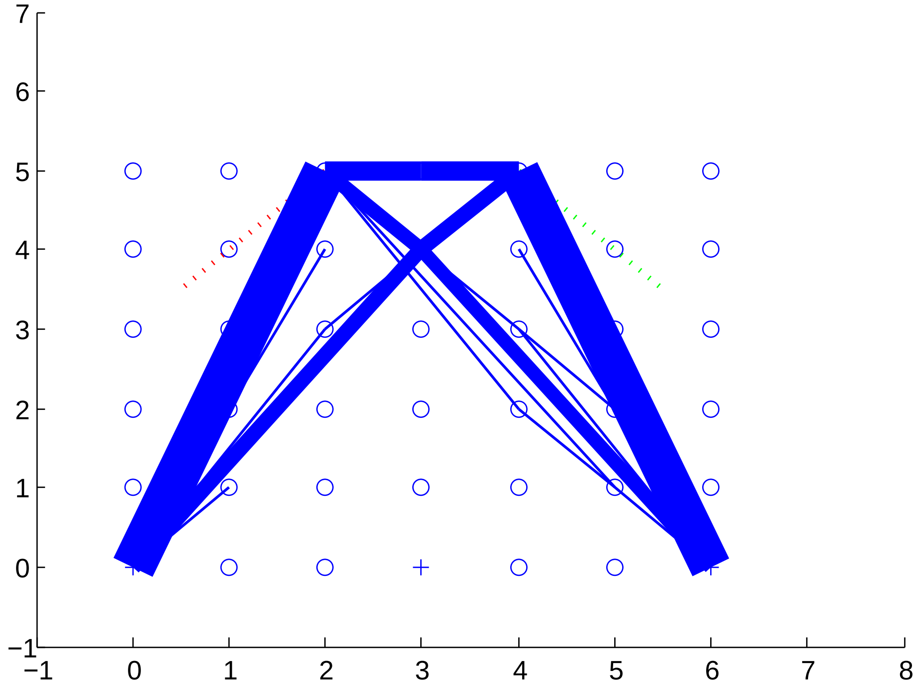
iteration number 600



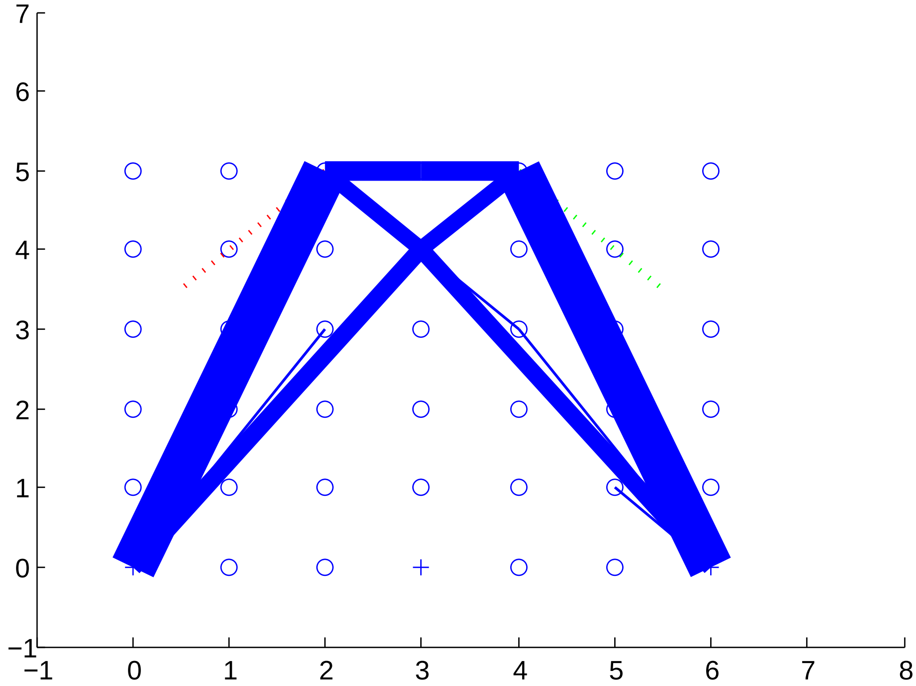
iteration number 800



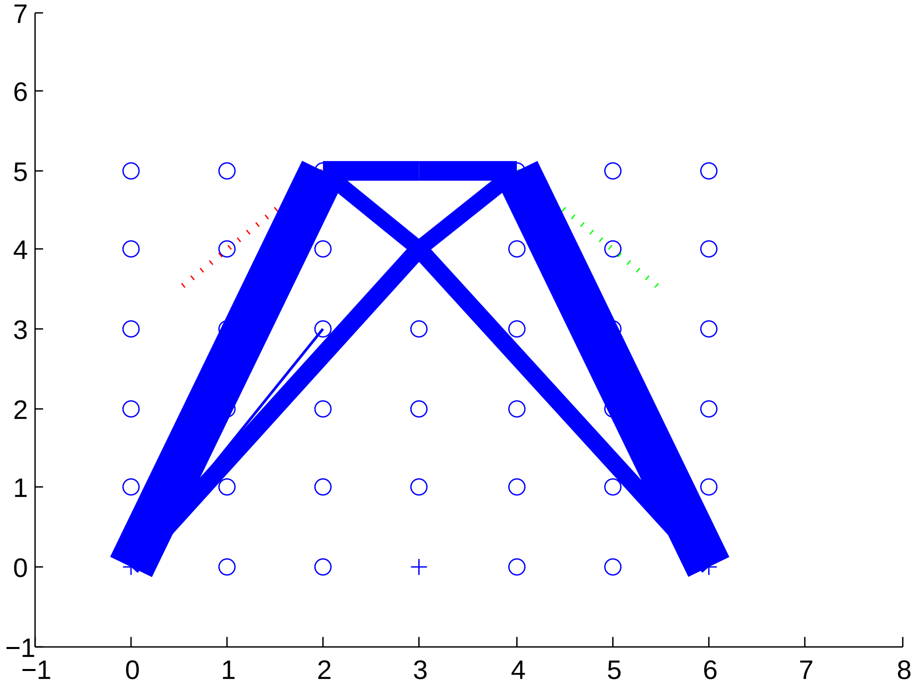
iteration number 1000



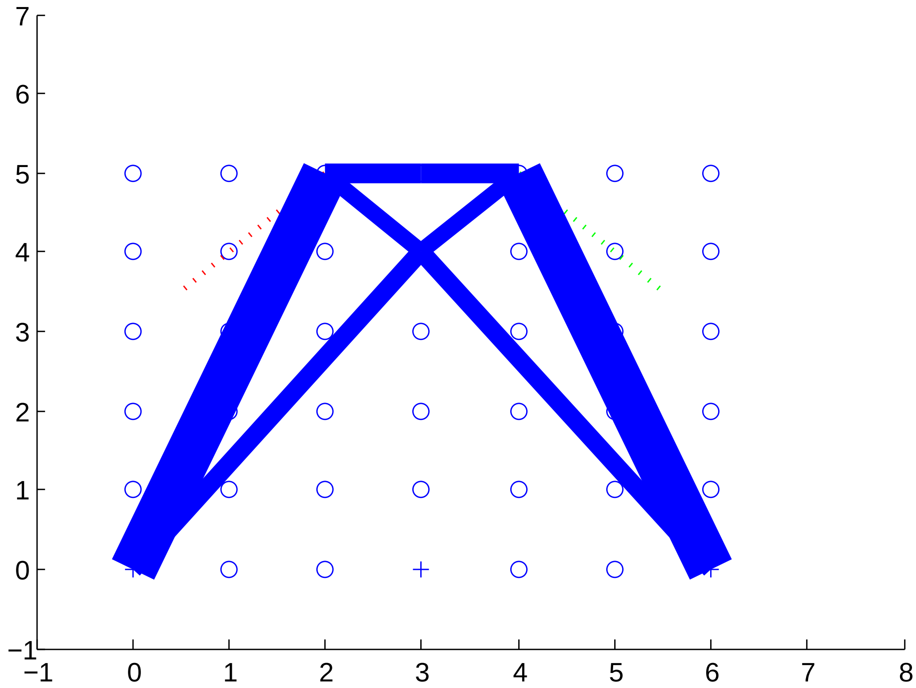
iteration number 2000



iteration number 3000

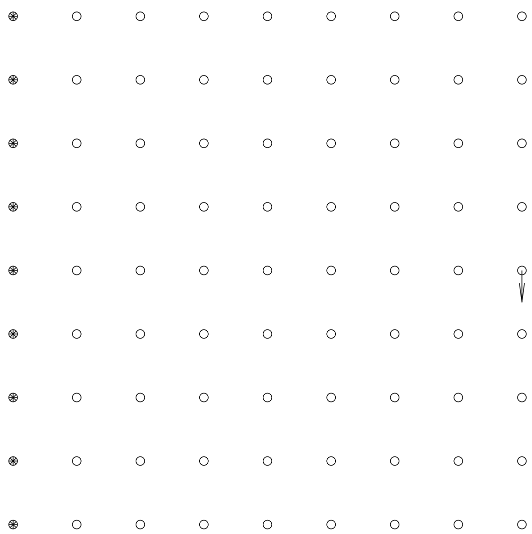


iteration number 8000



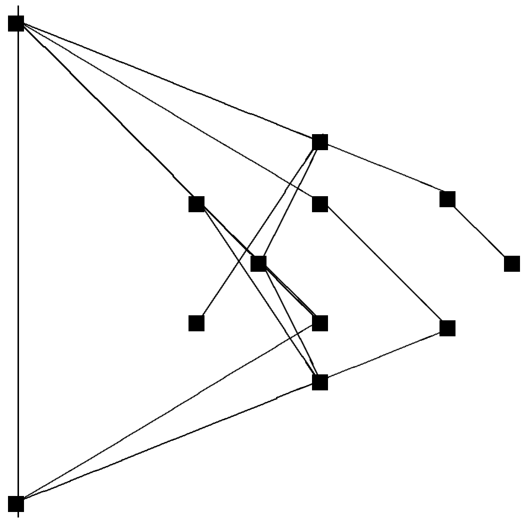
Can we trust the truss?

<u>Example:</u> **Assume we are designing a planar truss – a cantilever; the 9×9 nodal structure and the only load of interest f^* are as shown on the picture:**

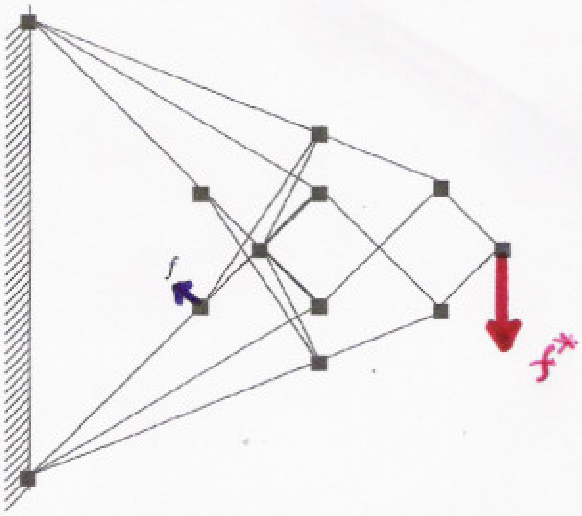


9×9 ground structure and the load of interest

The optimal single-load design yields a nice truss as follows:



Optimal cantilever (single-load design)
the compliance is 1.000



Optimal cantilever (single-load design)
the compliance is 1.000

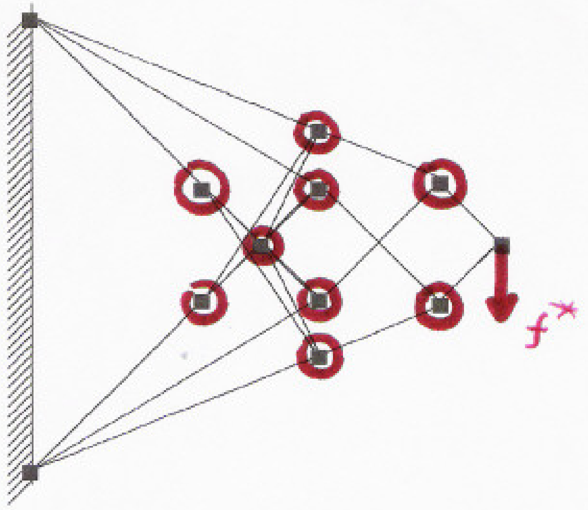
- It turns out that when the "load of interest" f^* is replaced with the small ($\| f \| = 0.005 \| f^* \|$) "badly placed" occasional load f, the compliance jumps from 1.000 to 8.4 (!)

- In order to improve the stability of the design, let us replace the single load of interest f^* by the ellipsoid containing this load and all loads f of magnitude $\| f \|$ not exceeding 10% of the magnitude of f^*:

$$\mathcal{F} = \{f = u_1 f^* + u_2 f_2 + f_3 u_3 + ... + u_{20} f_{20} \mid u^T u \le 1\},$$

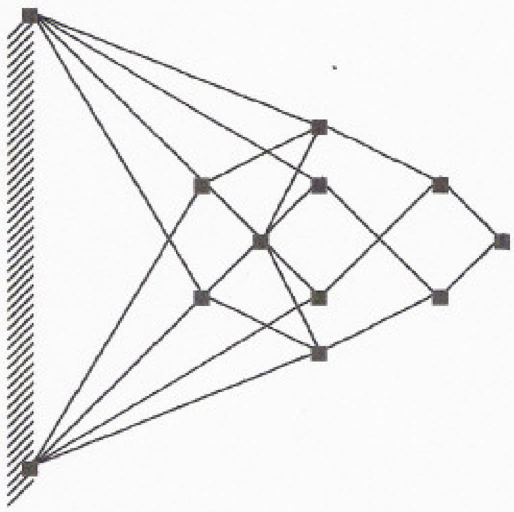
where $f_2, ..., f_{20}$ are of the norm $0.1 \| f^* \|$ and $f^*, f_2, ..., f_{20}$ is an orthogonal basis in $\mathbf{V} = \mathbf{R}^{20}$.

• **Passing from the single-load to the robust design, we modify the result as follows:**



Optimal cantilever (single-load design)

$\Downarrow$



"Robust" cantilever

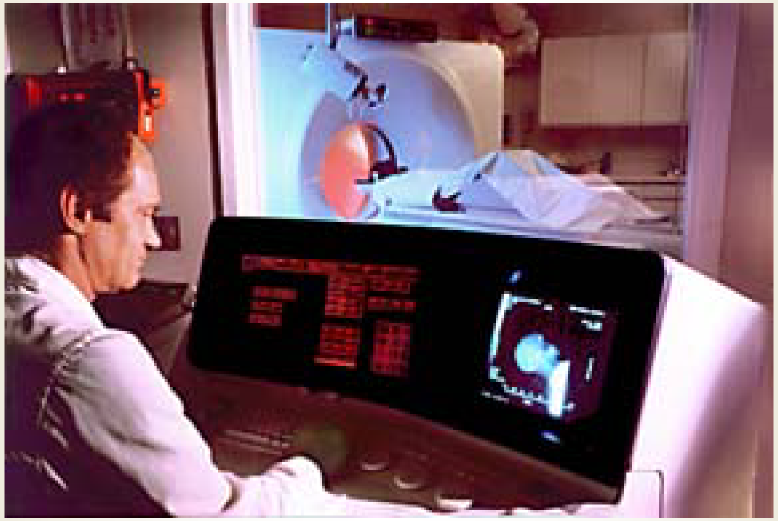
Compliances	**Design**	
	Single-load	**Robust**
Compliance w.r.t. f^*	**1.000**	**1.0024**
max **compliance w.r.t. loads** f**:** $\| f \| \leq 0.1 \| f^* \|$	**32000**	**1.03**

Convex Optimization in the Service of Medicine

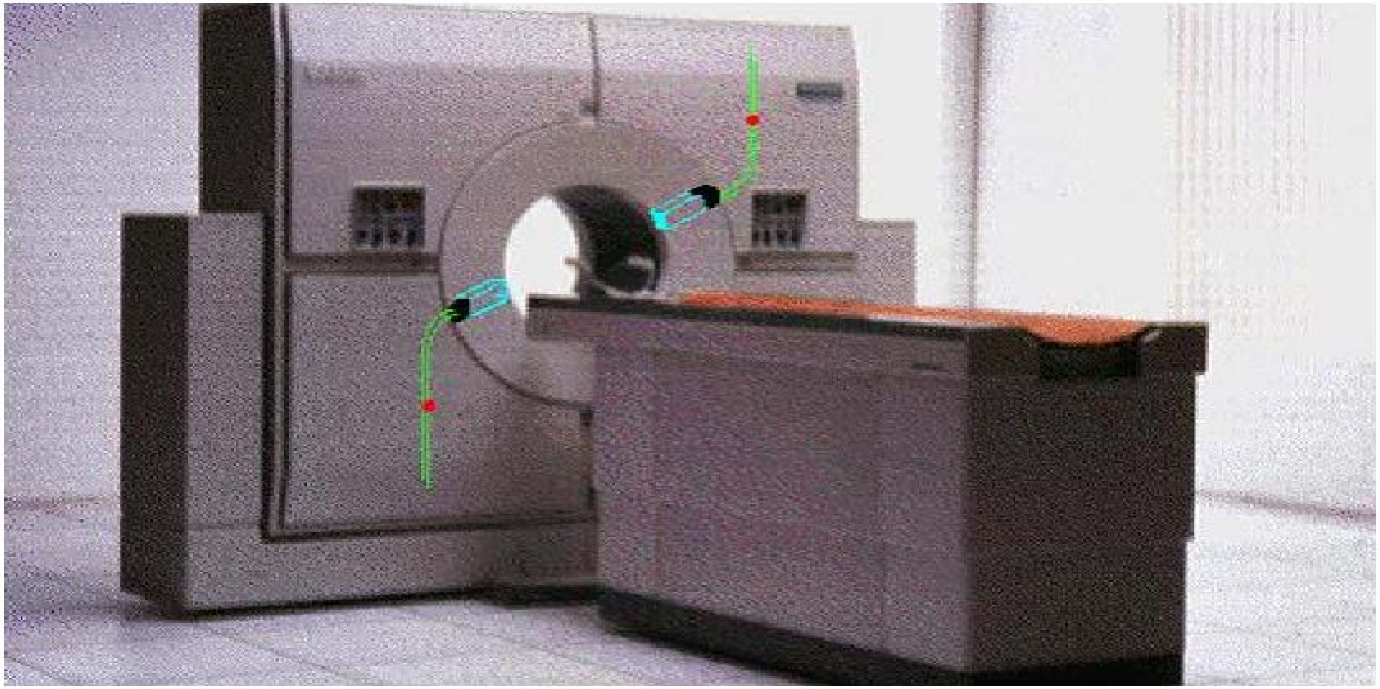
Basics

- In most cases, the location and extent of a disease is unknown. The first objective is to find efficient means of searching throughout the body to determine the exact location of the disease.
- Imaging is an extremely efficient process for accomplishing this aim, because data are presented in pictorial form.
- PET (Positron Emission Tomography) and SPET (Single Positron Emission Tomography) provide the means for imaging the rates of biologic processes in vivo.
- Imaging is accomplished through the integration of two technologies, the tracer kinetic assay method and computed tomography (CT).

Data Acquisition

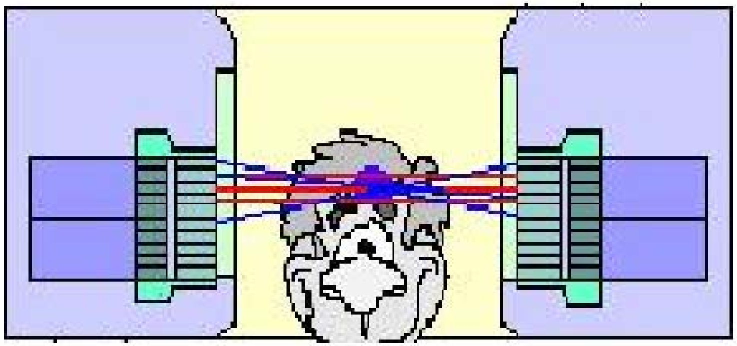


Application Example: 3D Imaging in Positron Emission Tomography



♣ PET is a powerful non-invasive medical diagnostic imaging technique for measuring the metabolic activity of cells in human body.
PET imaging is unique in that it shows the chemical functioning of organs and tissues, not just anatomic structures.

♣ In PET, patient is administered radioactive tracer. The tracer concentrates in the the desired areas (e.g., those of high metabolic activity).



When disintegrating, the tracer emits positrons which annihilate with near-by electrons and produce pairs of photons flying in opposite (completely random) directions at the speed of light. The photons nearly simultaneously hit two detectors of PET scanner. This event is registered, thus specifying a Line Of Response passing through the disintegration point.

♣ The data of PET reconstruction problem are given by (few millions of) LORs registered by the scanner, and the problem is to recover the distribution of the tracer.

♣ **An idealized mathematical model of the PET imaging problem is to recover a 2D (or 3D) density from its Radon transform — collection of integrals of the density along all lines in $\mathbf{R}^2$ (or $\mathbf{R}^3$).**

♠ **In reality, the Radon transform data registered by PET scanner are incomplete, noisy and discretized, which badly affects the quality of the Inverse Radon Transform imaging.**

♣ **Applying the Maximum Likelihood method, one ends up with the following convex optimization problem:**

$$\min_{\lambda}\left\{f(\lambda) = -\sum_{i=1}^{m} y_i \ln\left(\sum_{j=1}^{m} p_{ij}\lambda_j\right) : \lambda \geq 0, \sum_{i=1}^{n} \lambda_i \leq 1\right\} \qquad \textbf{(PET)}$$

- $\lambda \in \mathbf{R}^n$: **discretized tracer's density (design vector)**
- $y_i \geq 0$ – **# of LORs registered by i-th pair of detectors (data)**
- $p_{ij} \geq 0$ – **probability for LOR originating from j-th grid point to be registered by i-th pair of detectors (data)**

♣ **PET Imaging problems are extremely large-scale: in 3D,**

- **the design dimension n varies from 500,000 to 3,000,000**
- **the number m of log-terms in the objective varies from 3,000,000 to 25,000,000**

♠ **When solving typical** *nonlinear* **convex problems, the "price of accuracy digit" for all known polynomial time algorithms is as large as** $O(n^3)$**. With** $n \sim 10^5$**, this price is by six (!) orders of magnitude larger than the performance of modern computers (**~ 1 **Gfl/sec).**

$\Rightarrow$ **With known polynomial time methods, one cannot solve in a realistic time nonlinear convex problems with tens/hundreds of thousands design variables: just the very first iteration will last forever...**

Example: 3D Positron Emission Tomography Imaging by the "best fitting" IP method:

image resolution	n	CPU time per iteration (performance 1 Gfl/sec)
$64 \times 64 \times 64$	**262,144**	**2,5 hours**
$128 \times 128 \times 128$	**2,097,152**	$>$ **13 days**

Reconstruction Algorithms

MIRROR DESCENT METHOD
(Black-Box setting)

Problem

$$f_* = \min_{x \in X} f(x)$$

X convex compact set

f convex Lipschitz continuous on X:

$$|f(x) - f(y)| \leq L\|x - y\| \qquad \forall\, x, y \in X$$

f is given by a *first-order oracle* — a routine which, given $x \in X$, returns the value $f(x)$ and a subgradient $f'(x)$.

Iteration t given x_t, $f(x_t)$, $f'(x_t)$

$$x_{t+1} = \arg\min_{y \in X} \left\{ \ell_{x_t}(y) + \frac{1}{\gamma_t}\omega_{x_t}(y) \right\}$$

$$\begin{array}{rcll} \ell_{x_t}(y) & = & f(x_t) + (y - x_t)^T f'(x_t) & \text{linearization of } f \\ \omega_{x_t}(y) & = & \text{“distance”}(y, x_t) & \text{localizer} \\ 1/\gamma_t & = & \text{penalty parameter} & \end{array}$$

Classical ***Gradient Projection Method***:

$$\omega_x(y) = \|x - y\|_2^2$$

- Best method for $X = \{x \mid \|x\|_2 \leq 1\}$.

For X being a *simplex* :

$$X = \{x \mid \Sigma x_i = 1, \quad x \geq 0\}$$

(x_i are "probabilities") a classical distance function in statistical information theory, etc. is the *relative entropy*

$$\omega_x(y) = \Sigma y_i \log(y_i/x_i)\,.$$

With this choice, we get the MD algorithms for

$$\min_{x\in X} f(x) \qquad X = \text{simplex}.$$

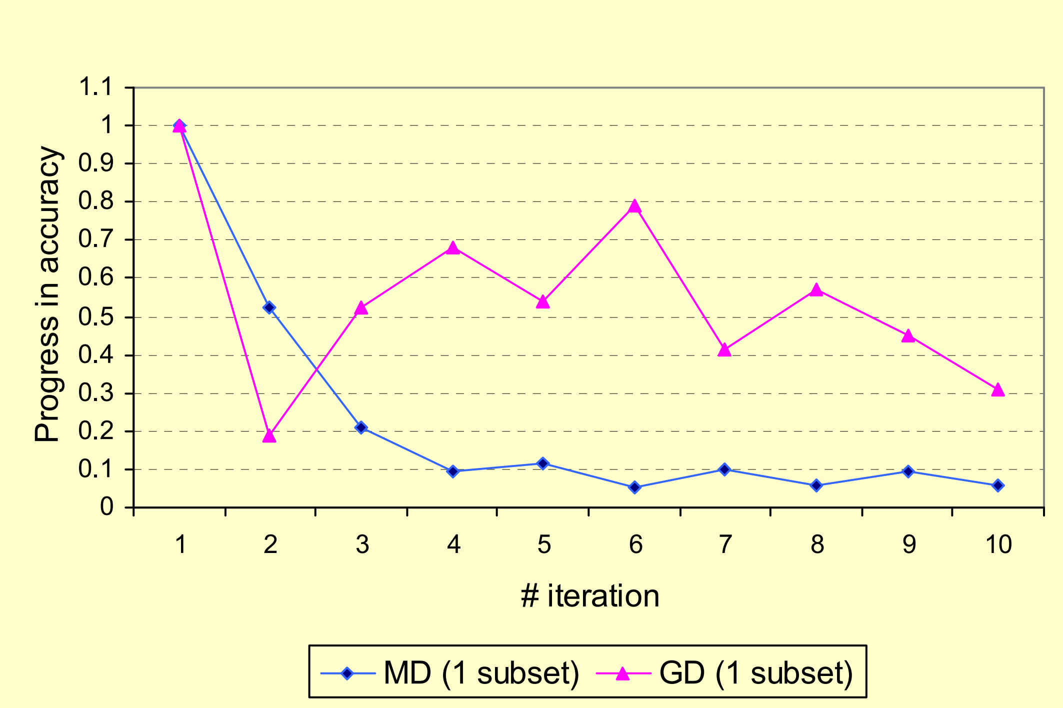
Theorem 1 *Let x^t be the best solution obtained up to iteration t. Then*

$$f(x^t) - f_* \leq 0(1)\, L_1(f)\sqrt{\log n} \cdot \frac{1}{\sqrt{t}}$$

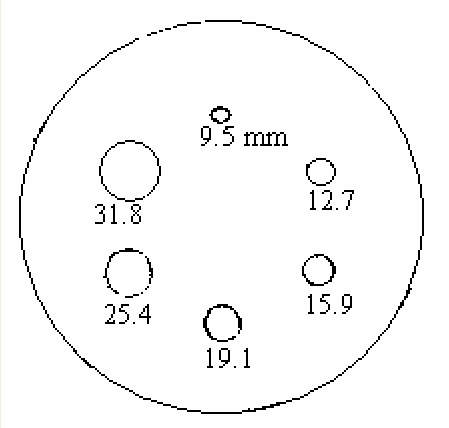
- efficiency estimate is essentially *dimension free*
- the MD algorithm is optimal for X = simplex
- similar result for $\min_{x \in X = \text{ spectahedron}} f(x)$

$$X = \{x \in S^n \mid x \succeq 0\,,\ \mathrm{Trace}(x) \leq 1\}$$

Hot Spheres Phantom (*n*=515,871)

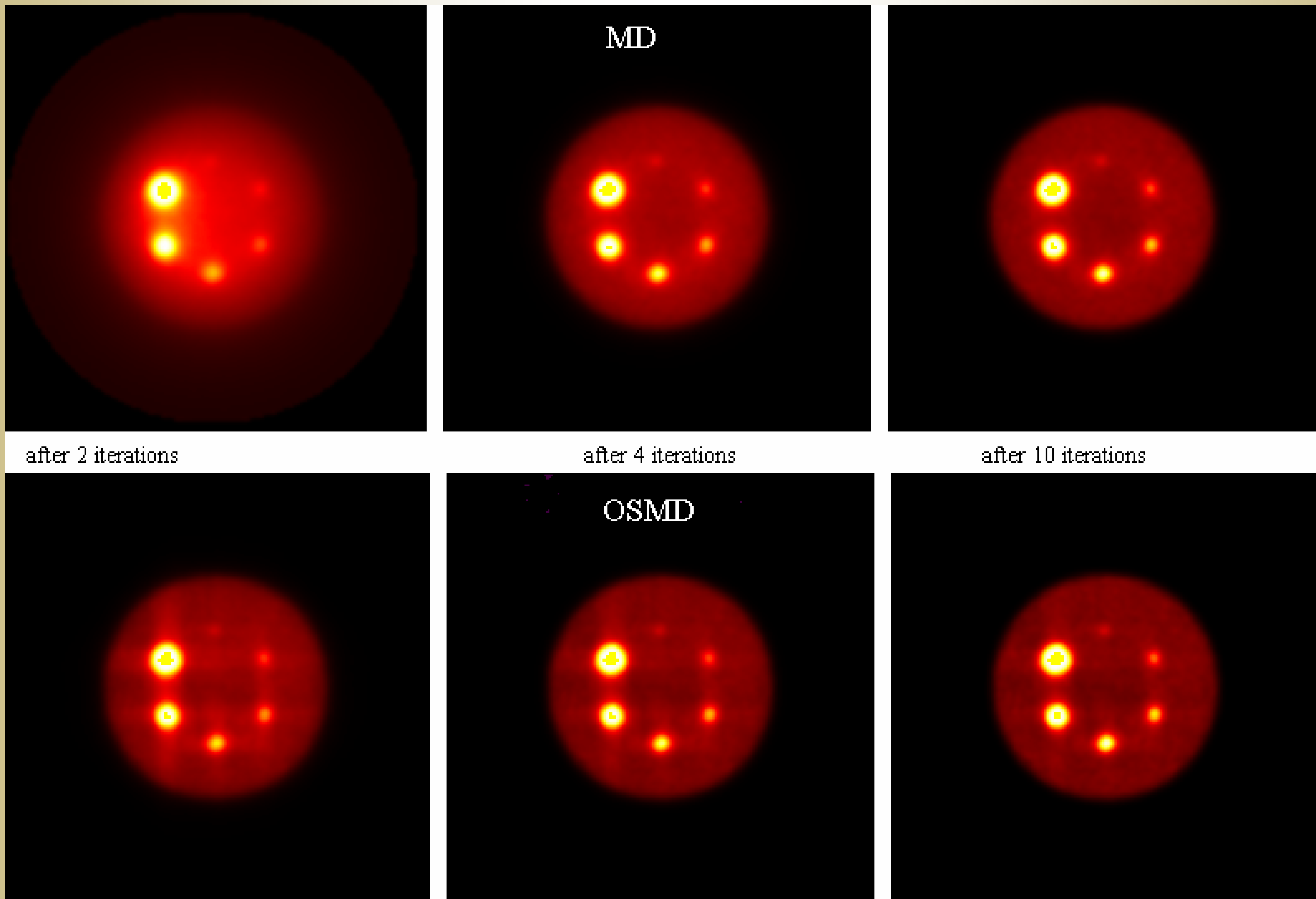


$$P_i = \frac{f_i - f_*}{f_1 - f_*}$$

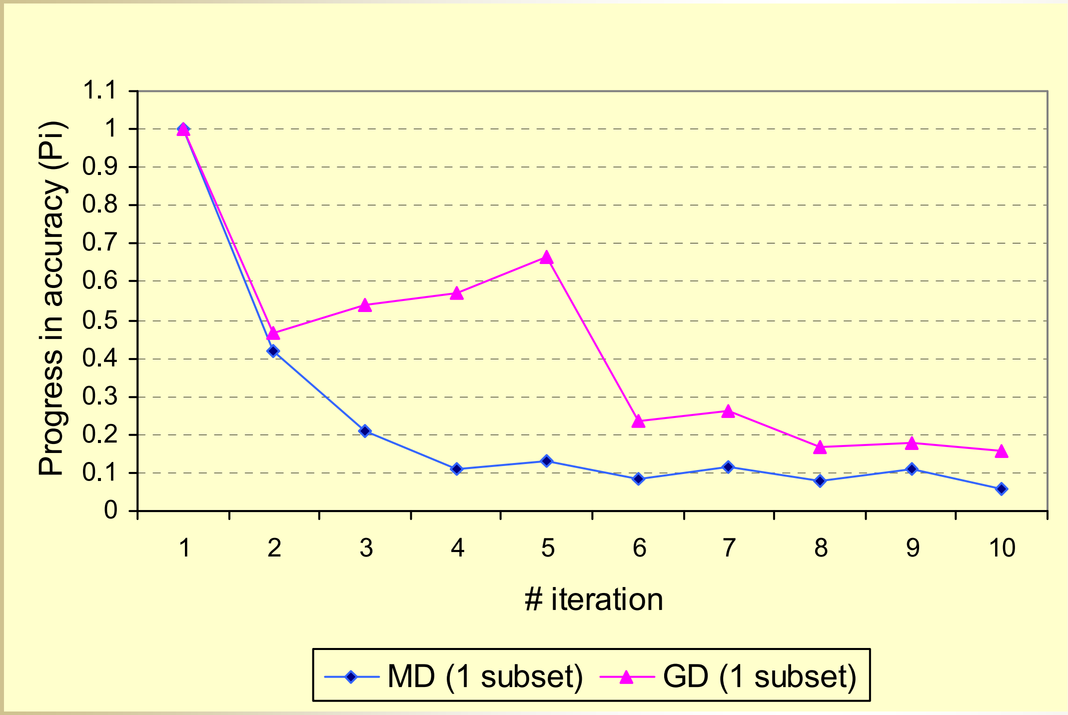


$$f_1 = -4.295\,\text{e}7;\quad f_{\text{bst}} = -5.230\,\text{e}7;\quad f_* \geq -5.283\,\text{e}7$$

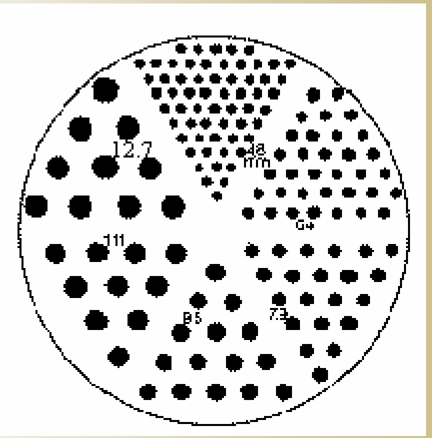
Hot Spheres (reconstruction by MD and OSMD)



Jaszczak Phantom $(n=515{,}871)$

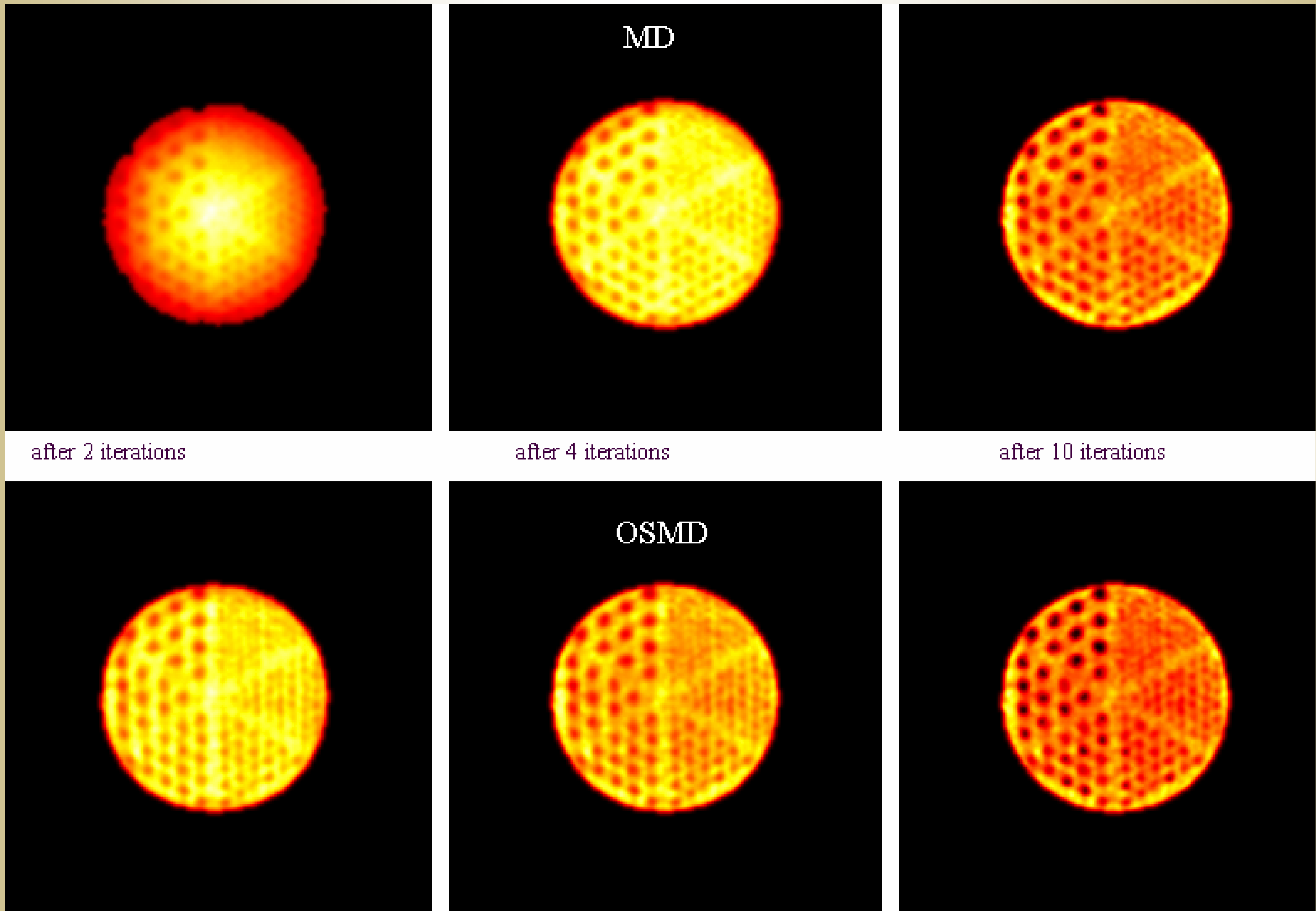


$$P_i = \frac{f_i - f_*}{f_1 - f_*}$$

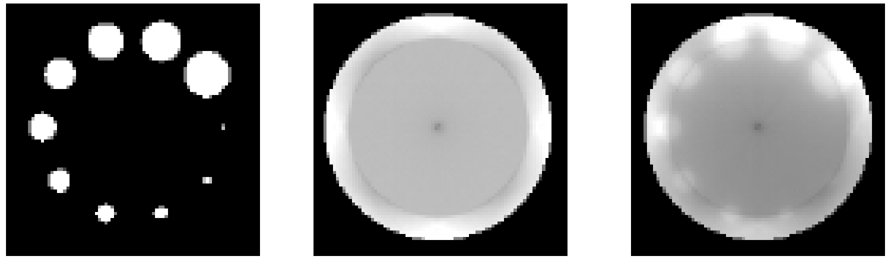


$$f_1 = -5.022\mathrm{e}7;\ \ f_{\mathrm{bst}} = -6.030\mathrm{e}7;\ \ f_* \geq -6.094\mathrm{e}7$$

Jaszczak Phantom (reconstruction by MD *and* OSMD*)*



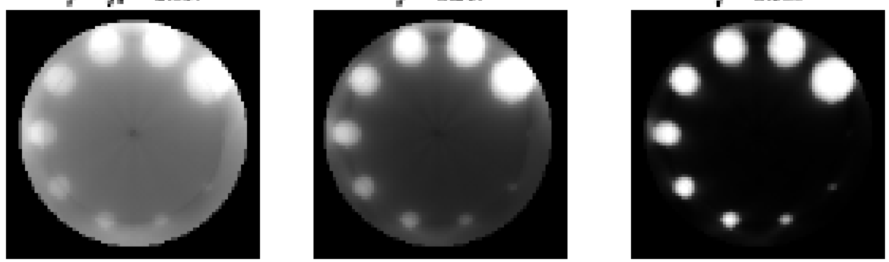
♣ **Experiment 1: noiseless measurements (brighter image correspond to higher tracer's density):**



True image: 10 "hot spots" $f = f_* = 2.817$

$x^1 = n^{-1}(1,\dots,1)^T$ $f = 3.247$

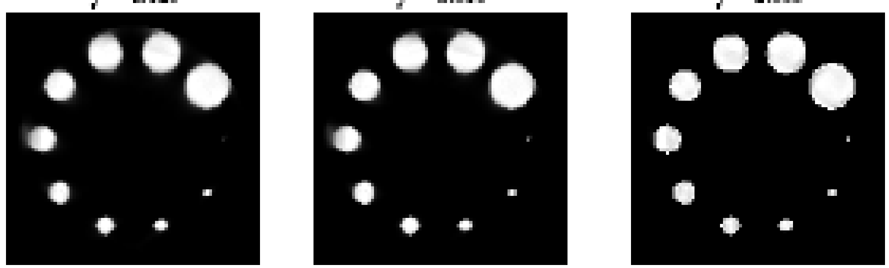
x^2 – some traces of 8 spots $f = 3.185$



x^3 – traces of 8 spots $f = 3.126$

x^4 – some traces of 9-th spot $f = 3.016$

x^8 – 10-th spot still missing... $f = 2.869$

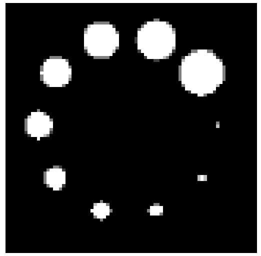


x^{24} – trace of 10-th spot $f = 2.828$

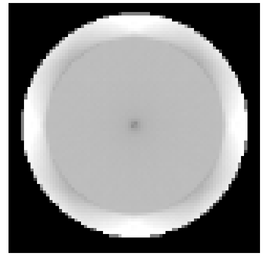
x^{27} – all 10 spots in place $f = 2.823$

x^{31} – that is it... $f = 2.818$

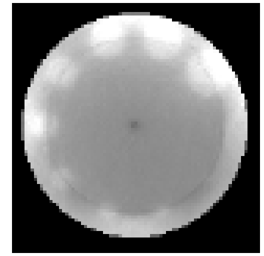
♣ Experiment 2: noisy measurements (at average, 40 LOR's per bright pixel, 63,092 LOR's totally):



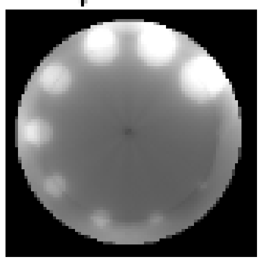
True image: 10 "hot spots"
$f = -0.883$



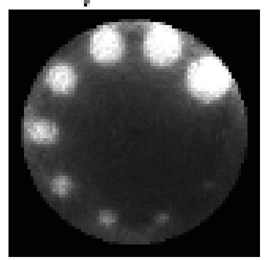
$x^1 = n^{-1}(1,\ldots,1)^T$
$f = -0.452$



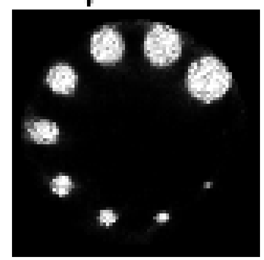
x^2 – light traces of 5 spots
$f = -0.820$



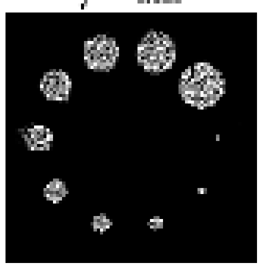
x^3 – traces of 8 spots
$f = -0.855$



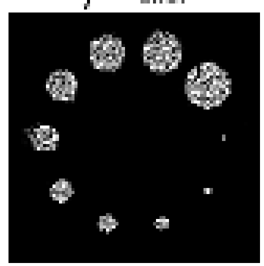
x^4 – 8 spots in place
$f = -0.707$



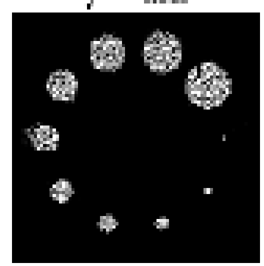
x^8 – 10th spot still missing...
$f = -0.865$



x^{12} – all 10 spots in place
$f = -0.872$

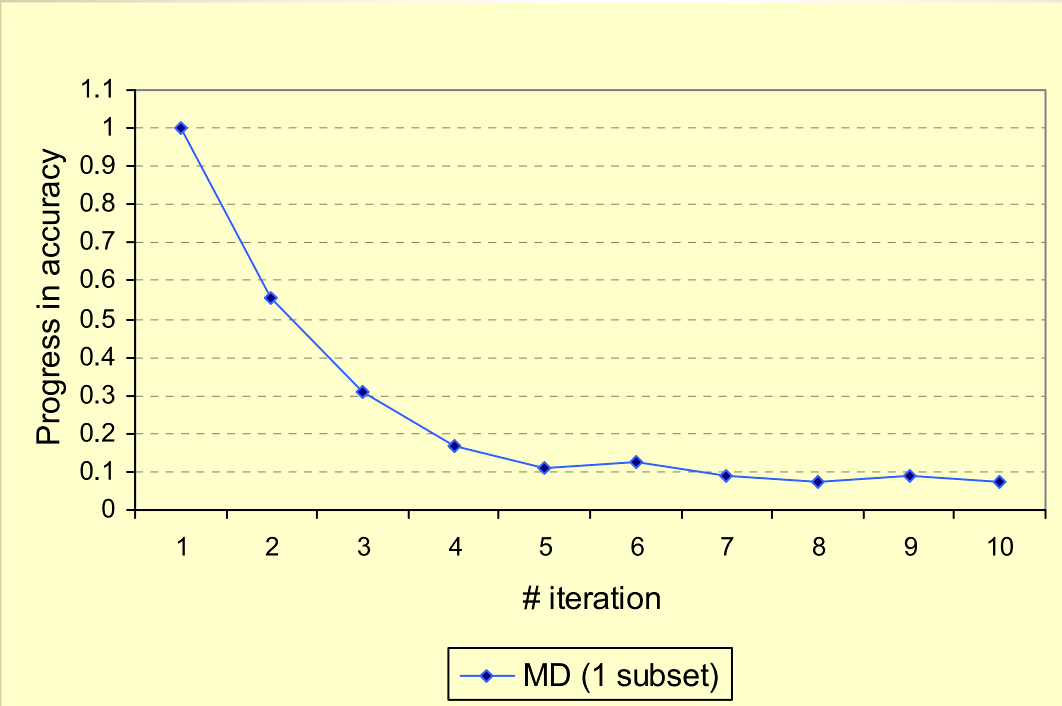


x^{31} – all 10 spots in place
$f = -0.886$



x^{40} – ...
$f = -0.896$

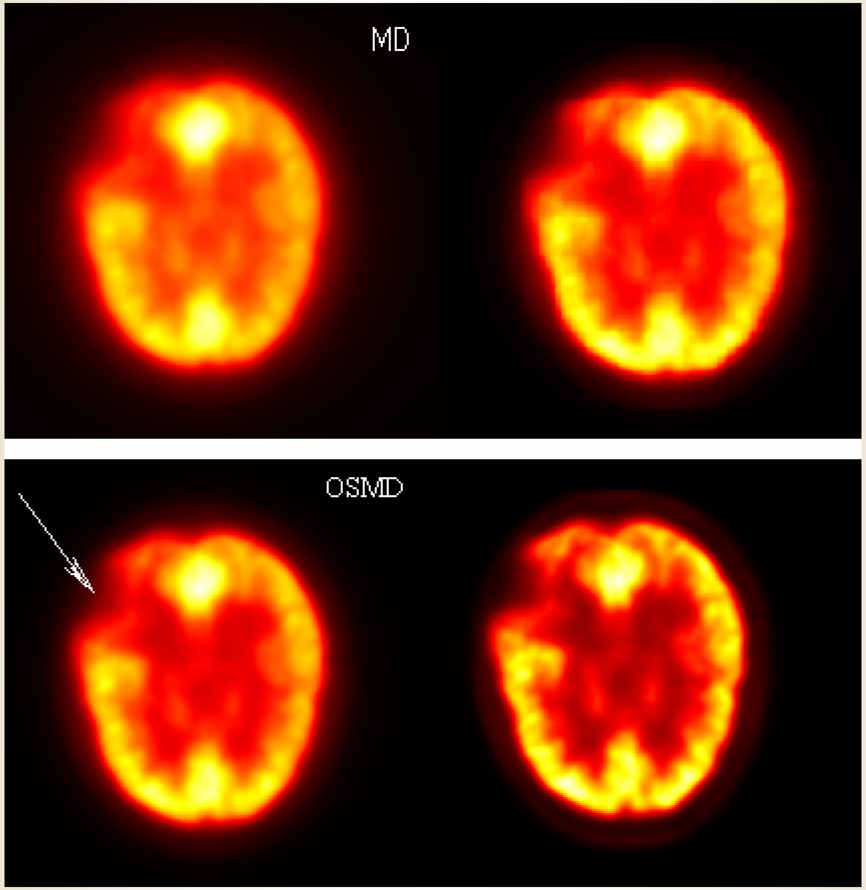
Brain study -clinical (reconstruction by MD *and* OSMD*)*
GE Advance Tomograph, n=2,763,635, # bins=25,000,000



$$P_i = \frac{f_i - f_*}{f_1 - f_*}$$

$$f_1 = -1.463\text{e}9; \quad f_{\text{bst}} = -2.009\text{e}9; \quad f_* \geq -2.050\text{e}9$$

Brain study -clinical (reconstruction by MD *and* OSMD*)*
GE Advance Tomograph, n=2,763,635, # bins=25,000,000



Optimization in Flight...

Application Example: Free Material Optimization

♣ FMO is a methodology for design of mechanical structures. In FMO, one seeks how to distribute a given amount of elastic material over a given domain in order to get a structure capable to withstand best of all a given collection of external loads. It is assumed that

- the mechanical properties of the material (its rigidity tensor) may vary, in an arbitrary fashion, from point to point;
- the rigidity of a construction w.r.t. a given external load is measured by the compliance – potential energy capacitated by the construction at the static equilibrium corresponding to the load;

♠ The goal is, given the weight of the construction, to minimize its largest, over a given set of loading scenarios, compliance.

♣ Usually it is technically impossible or too expensive to implement an FMO design "as it is". The role of FMO is in providing a good guess for the structure of the would-be construction. After the structure is guessed, the construction is designed from traditional materials via standard engineering techniques.

♣ **With Finite Element discretization, the Multi-Load FMO problem is**

$$\min_t \left\{ \max_{\ell=1,...,K} f_\ell^T S^{-1}(t) f_\ell : t_i \succeq 0, \sum_i \mathrm{Tr}(t_i) \leq 1 \right\} \qquad \textbf{(FMO)}$$

where

- t_i, $i = 1, ..., N$, **are symmetric** 3×3 **(in 2D) or** 6×6 **(in 3D) variable matrices (rigidity tensors of the material in Finite Element cells),**
- f_ℓ, $\ell = 1, ..., K$, **are** M**-dimensional data vectors representing loading scenarios,**
- $S(t) = \sum\limits_{i,s} b_{is}^T t_i b_{is}$ **is the** $M \times M$ **stiffness matrix of the construction.**

♣ **In a realistic 2D FMO problem,**

- **the number** N **of Finite Element cells is tens of thousands**

$\Rightarrow$ **design dimension of (FMO) is of order of 50,000 – 200,000**

- **the size** M **of the stiffness matrix is** $\approx 2N$

$\Rightarrow$ **it is a nontrivial problem just to compute the objective!**

IMPROVING MD

♣ Utilizing past information. A severe practical disadvantage of SD/MD is that they are memoryless: they use the first-order information only from the last step. Free of this drawback modifications of Subgradient Descent – bundle methods – are well-known and widely used.

♠ Recently, the first bundle version of Mirror Descent was developed (Non-Euclidean Restricted Memory Level method, Ben-Tal and Nemirovski 2004). The NERML algorithm

- exhibits the same theoretical rates of convergence as the MD
- in practice, provides full control of how the accumulated information is utilized and thus allows for tradeoff between iteration complexity and convergence rate.

The FMO problem is to minimize a convex function $F : S^n \to R$ over a spectahedron X

$$\min\{F(t) \mid t \in S^n,\ t \succeq 0,\ Tr(t) \leq 1\}$$

Here, when employing the MD/NERML algorithms, the choice of the "distance function" is based on the "entropy" function $\omega : S^n \to R$ given by

$$\omega(S) = Tr(S + \sigma I) \log(S + \sigma I), \quad \sigma = \delta/n$$

The major step in the MD/NERML algorithms at each iteration is to solve a problem of the following type

$$\min\{\omega(S) + P^T S\}$$

which is here

$$\min_{S \in X}\{Tr(S + \sigma I) \log(S + \sigma I) + Tr(PS)\} \tag{1}$$

Let U be an orthogonal matrix diagonalizing P, i.e.

$$P \equiv U d U^T \qquad d = \text{diagonal}$$

Substitute for S a new matrix variable

$$y = U S U^T$$

problem (1) becomes

$$\min_{y \in X} \{ Tr(y + \sigma I) \log(y + \sigma I) + Tr(dy) \} \tag{2}$$

This function depends only on the eigenvalues of $y + \sigma I$

diagonal

$$\min_{y \in X} \{Tr(y + \sigma I) \log(y + \sigma I) + Tr(dy)\} \tag{2}$$

Let J be a matrix $J = \mathrm{diag}(\pm 1)$. Then

$$y + \sigma I \quad \text{and} \quad J(y + \sigma I)J = JyJ + \sigma I$$

all have the same eigenvalues.

Also, since d is diagonal:

$$Tr(dy) = Tr(JdJy) = Tr(dJyJ)$$

It follows that replacing y in the objective function of (2) with JyJ, we get the same value (INVARIANCE)

Conclusion I If y^* solves (2), then Jy^*J also solves (2) for any matrix $J = \mathrm{diag}(\pm 1)$.

Conclusion II The average overall 2^n matrices of type J of the solution Jy^*J, i.e.

$$(1/2^n) \sum (Jy^*J)$$

is also an optimal solution. **BUT**, this latter matrix is exactly

$$\mathrm{diag}(y^*)$$

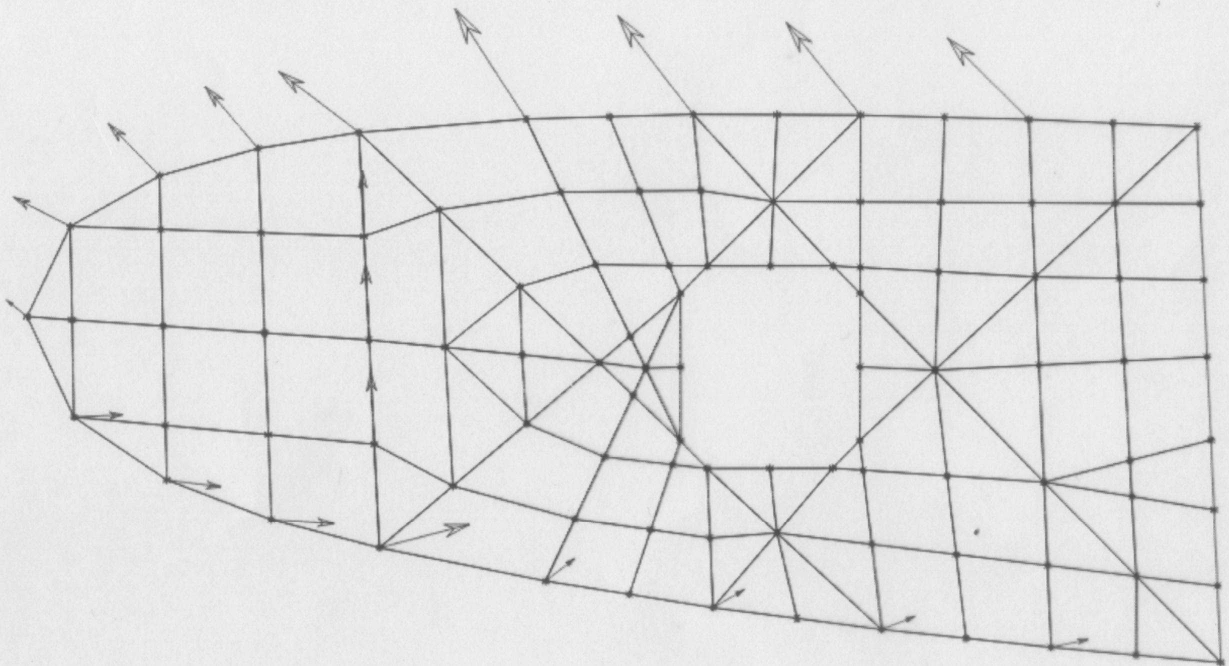
Conclusion III Problem (1) reduces to (2) which reduces to:

$$\left[\begin{array}{ll} \min\limits_{\zeta \in R^n} & \sum (\zeta_i + \sigma) \log(\zeta_i + \sigma) + d_i \zeta_i \\ s.t. & \sum \zeta_i \leq 1 \\ & \zeta_i \geq 0 \end{array}\right. \tag{3}$$

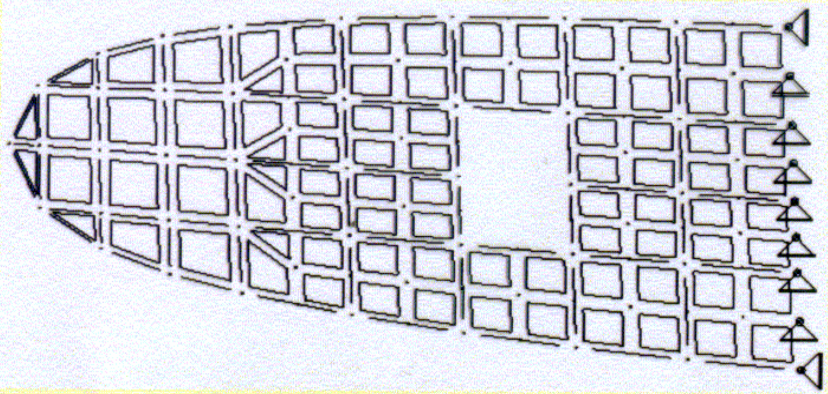
This problem can be solved (almost) explicitly!

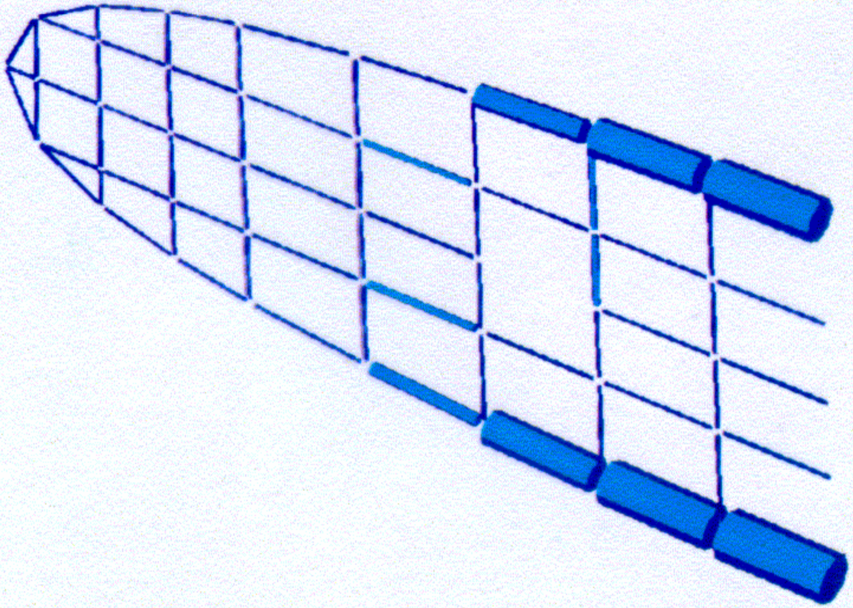
Design of stiffeners: MOPED & MBB-LAGRANGE

Design of stiffeners: MOPED & MBB-LAGRANGE

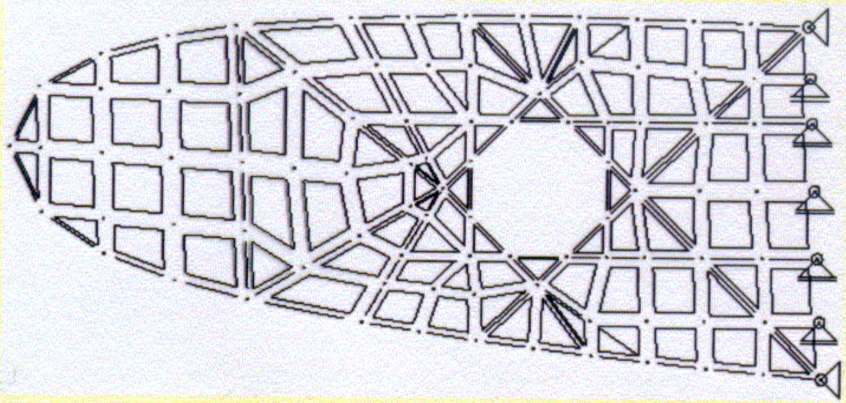


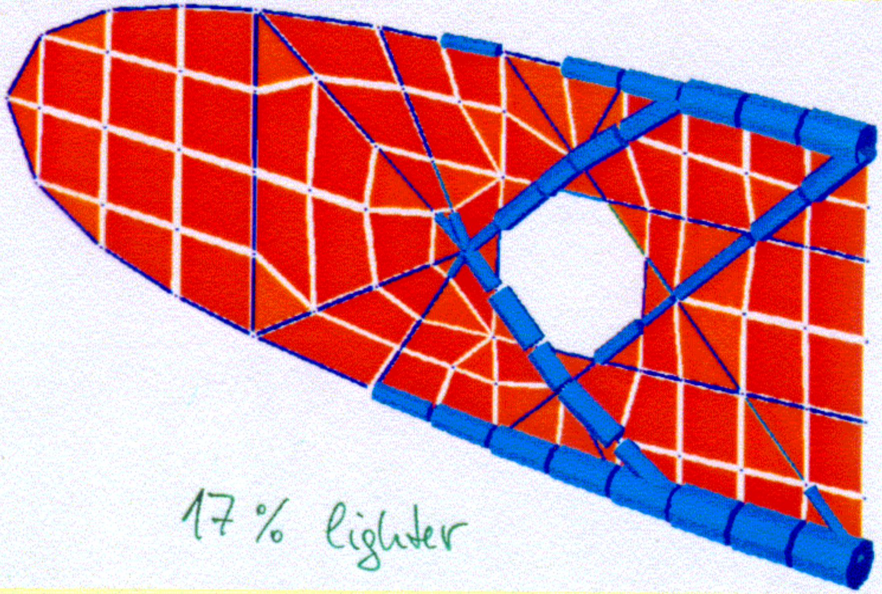
Reference design

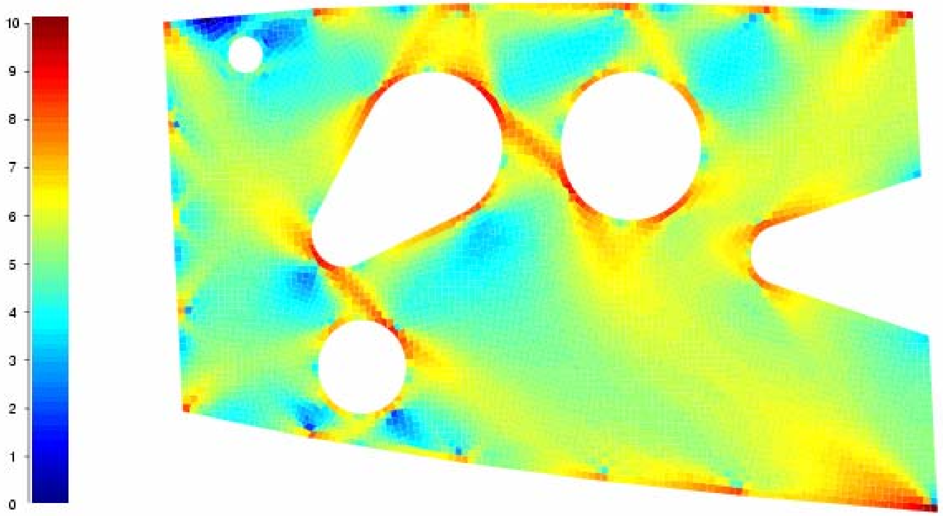




FMO based design







Wing element for Aerobus A380
FMO design by NERML
80 iterations

Implementation, Erlangen
University and European Aero
Defence and Space Co.

$n = 39,780, \ N = 6,630, \ M = 13,824$

Free Material Optimization: element of aircraft wing
FMO allowed for 17% reduction in element's weight

Based on computational results for maximum stiffness and quite a bit of engineering interpretation a new type of structure was devised for the ribs which gave a weight benefit against traditional and competitive honeycomb/ composite designs (up to 40% !)

A total weight saving of more than 500 kg per wing was obtained by optimizing the ribs in the area shown. These are now — since April 27, 2005 - the first topology optimized parts in flight.